New York City
Interscholastic
Mathematics
League

Sophomore-Freshman Division

CONTEST NUMBER 1

PART I *FALL, 2006* *CONTEST 1* *TIME: 10 MINUTES*

F06SF1 Compute the smallest perfect square integer, greater than zero that is divisible by 4, 6, and 7.

F06SF2 A circle is inscribed in a triangle with sides of length 9, 12, and 15. Compute the radius of the circle.

PART II *FALL, 2006* *CONTEST 1* *TIME: 10 MINUTES*

F06SF3 Compute the number of ordered pairs of positive integers (x, y) such that: $x^2 + y^2 \leq 50$.

F06SF4 Mr. Camel's math class has 25 students who take a test. If passing is 65 or above and the average of the passing grades is 82, the average of the failing grades is 57, and the average of all of the grades is 76. Compute the number of students who passed the test.

PART III *FALL, 2006* *CONTEST 1* *TIME: 10 MINUTES*

F06SF5 Compute the remainder when 2^{2006} is divided by 7.

F06SF6 The medians to the legs of right triangle ABC measure 5 and $\sqrt{40}$. Compute the length of the hypotenuse of triangle ABC.

ANSWERS:

F06SF1	**1764**
F06SF2	**3**
F06SF3	**33**
F06SF4	**19**
F06SF5	**4**
F06SF6	**$2\sqrt{13}$**

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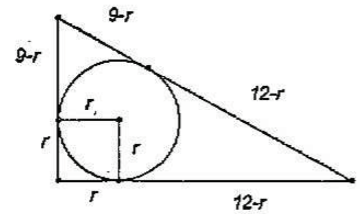
F06SF1 The answer will be the prime factors to the smallest possible even powers. $2^2 \cdot 3^2 \cdot 7^2 = 1764$.

F06SF2 By the Pythagorean Theorem, the triangle is a right triangle. From the diagram, we get: $9-r+12-r=15 \rightarrow r=3$. OR, we can draw line segments from the center of the circle to each of the vertices. The area of the right triangle is $\frac{1}{2}\cdot 9\cdot 12=54 \rightarrow \frac{1}{2}r\cdot 9+\frac{1}{2}r\cdot 12+\frac{1}{2}r\cdot 15=54 \rightarrow r=3$.



F06SF3 Listing them, we get: from (1, 1) to (1, 7), from (2, 1) to (2, 6), from (3, 1) to (3, 6), from (4, 1) to (4, 5), from (5, 1) to (5, 5), from (6, 1) to (6, 3), and (7, 1). $7+6+6+5+5+3+1=33$.

F06SF4 Let the number of students passing the test be x. The sum of the grades is:

$$82x+57(25-x)=76\cdot 25$$
$$25x+1425=1900 \rightarrow x=19.$$

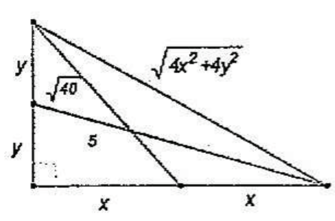
F06SF5 $2^{2006}=\left(2^3\right)^{668}\cdot 2^2$. When $2^3=8$ is divided by 7, the remainder is 1. Thus the answer is $2^2=4$.

OR

By modular arithmetic:

$$2^{2006} \equiv x \bmod 7$$
$$\left(2^3\right)^{668}\cdot 2^2 \equiv x \bmod 7$$
$$(8)^{668}\cdot 2^2 \equiv x \bmod 7$$
$$(1)^{668}\cdot 2^2 \equiv x \bmod 7$$
$$2^2 \equiv x \bmod 7$$
$$x=4.$$

F06SF6 From the diagram: $\begin{cases} x^2+4y^2=40 \\ 4x^2+y^2=25 \end{cases}$. Thus we get: $5x^2+5y^2=65 \rightarrow 4x^2+4y^2=52 \rightarrow \sqrt{4x^2+4y^2}=\sqrt{52}=\mathbf{2\sqrt{13}}$.



SOPHOMORE-FRESHMAN DIVISION CONTEST NUMBER 2

PART I *FALL, 2006* *CONTEST 2* *TIME: 10 MINUTES*

F06SF7 Ming has a penny, a nickel, a dime, a quarter, a half dollar and a silver dollar. Compute the number of possible different amounts of money he could spend by using one or more of these coins.

F06SF8 The number 9991 can be written as the product of two primes. Compute the primes.

PART II *FALL, 2006* *CONTEST 2* *TIME: 10 MINUTES*

F06SF9 Two numbers have a sum of 8 and a product of 11. Compute the sum of the reciprocals of the numbers.

F06SF10 Two roots of the equation $x^3+ax^2+17x+b=0$ are 1 and 2. Compute the third root.

PART III *FALL, 2006* *CONTEST 2* *TIME: 10 MINUTES*

F06SF11 Compute the number of ordered pairs of positive integers (x,y) such that: $3x+5y=500$.

F06SF12 If p and q are the roots of the equation $2x^2+3x+7=0$, compute pq^3+qp^3.

ANSWERS:

F06SF7	**63**
F06SF8	**103, 97**
F06SF9	$\frac{8}{11}$
F06SF10	**5**
F06SF11	**33**
F06SF12	$\frac{-133}{8}$

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F06SF7 For each coin, Ming can either choose it or not. Thus there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$ possibilities. One of these, choosing no coins, we must exclude, thus we get **63**.

OR: A set of n items has 2^n subsets. Continue as above.

F06SF8 $9991 = 10000 - 9 = (100+3)(100-3) = 103 \cdot 97$. We can now verify that these are primes (although the problem states that they are) and the answers are **103, 97**.

F06SF9 Let the numbers be x and y. $\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \mathbf{\frac{8}{11}}$.

F06SF10 The sum of the products of the roots taken in pairs is 17. $1 \cdot r + 2 \cdot r + 1 \cdot 2 = 17 \rightarrow r = \mathbf{5}$.
OR:
Substituting the given roots in the equation we get $a + b = -18$ and $4a + b = -42 \rightarrow a = -8$. In the given equation the sum of the roots is $\frac{-a}{1} = -a \rightarrow 1 + 2 + r = 8 \rightarrow r = 5$.

F06SF11 Solving for y, we get: $y = \frac{500-3x}{5}$. y will be a positive integer if x is 5, 10, 15, …,165. This is 33 numbers, thus there are **33** pairs.

F06SF12 $pq^3 + qp^3 = pq(p^2 + q^2) = pq\left((p+q)^2 - 2pq\right)$. By the sum and product of the roots formulas, $pq = \frac{7}{2}, p + q = -\frac{3}{2}$. $\frac{7}{2}\left(\left(-\frac{3}{2}\right)^2 - 7\right) = \frac{7}{2} \cdot \left(\frac{-19}{4}\right) = \mathbf{\frac{-133}{8}}$.

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SOPHOMORE-FRESHMAN DIVISION CONTEST NUMBER 3

PART I *FALL, 2006* *CONTEST 3* *TIME: 10 MINUTES*

F06SF13 Compute the number of even four-digit integers greater than 3000 that can be made using the digits 2, 3, 4, and 7, without repetition.

F06SF14 Mickey, working alone, can build a model plane in 5 hours. Minnie, working alone, can build the same model plane in 4 hours. If Mickey works alone for 2 hours, and then Minnie helps him, compute the number of **minutes** it will take for them to complete the job working together.

PART II *FALL, 2006* *CONTEST 3* *TIME: 10 MINUTES*

F06SF15 A convex polygon has 35 diagonals. Compute the number of vertices.

F06SF16 In a Fibonacci type sequence of increasing positive integers, each number after the first two is equal to the sum of the two numbers that immediately precede it. If the tenth number is 322, compute the fifth number.

PART III *FALL, 2006* *CONTEST 3* *TIME: 10 MINUTES*

F06SF17 Compute x: $x+\sqrt{x-2}=4$.

F06SF18 Four men and four women are seated at a circular table with eight chairs and they sit man, woman, man, woman, etc. Compute the number of ways they can be seated. (Note that if everyone moves one chair clockwise, this is not considered a new arrangement, as we do not care which seat they have, only their relative position with respect to each other.)

ANSWERS:

F06SF13	**10**
F06SF14	**80**
F06SF15	**10**
F06SF16	**29**
F06SF17	**3**
F06SF18	**144**

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F06SF13 The first digit must be 3, 4, or 7. If 4 is the first digit, 2 must be the last digit and there are two ways to do this. If 3 or 7 is the first digit, then 2 or 4 must be the last digit and there are four ways for each. Thus we have $2+4+4=\mathbf{10}$.

F06SF14 Since Mickey worked alone for 2 hours, there is $\frac{3}{5}$ of the job to complete when they work together. In one hour, Mickey does $\frac{1}{5}$ of the job and Minnie does $\frac{1}{4}$ of the job.

$\frac{1}{5}+\frac{1}{4}=\frac{\frac{3}{5}}{x}\rightarrow\frac{9}{20}=\frac{3}{5x}\rightarrow 45x=60\rightarrow x=\frac{4}{3}\,hours$. Since the question asked for the number of minutes, the answer is **80** minutes.

F06SF15 In a convex polygon, we can draw a diagonal from any vertex to another, except for the two adjacent vertices and the vertex itself. Thus for an *n-gon* we can draw $(n-3)$ diagonals from n vertices. We must be careful not to count a diagonal twice, so the total number of diagonals is $\frac{n}{2}\cdot(n-3)=35\rightarrow n^2-3n-70=0\rightarrow n=\mathbf{10}$.

F06SF16 The numbers in the sequence can be represented as $a_1, a_2, a_1+a_2, a_1+2a_2, 2a_1+3a_2, 3a_1+5a_2, 5a_1+8a_2, 8a_1+13a_2, 13a_1+21a_2, 21a_1+34a_2$. Now we have $21a_1+34a_2=322$. Since $21a_1$ and 322 are divisible by 7, and 34 is not divisible by 7, a_2 must be divisible by 7, thus it is 7, since anything else would be too large. Thus: $a_2=7, a_1=4\rightarrow 2a_1+3a_2=\mathbf{29}$.

F06SF17 $\sqrt{x-2}=4-x\rightarrow x-2=16-8x+x^2\rightarrow x^2-9x+18=0\rightarrow x=3,6$. Checking the answers, we reject the 6 and the answer is **3**.

F06SF18 Let the first man sit down. Now there are 3! = 6 ways for the other men to sit. There are now 4! = 24 ways for the women to sit. There are thus 6 x 24 = **144** arrangements.