New York City Interscholastic Math League
Soph-Frosh Division   Contest Number 1

Part I   Spring 2011   Contest 1   Time: 10 Minutes

S11S1 Compute the real value of x so that $|x - 2.5| = |x - 3.5|$.

S11S2 Integers a, b, c, d are randomly chosen from 0 to 101, inclusive. They are chosen with replacement. Compute the probability that $|ab - cd|$ is odd.

Part II   Spring 2011   Contest 1   Time: 10 Minutes

S11S3 Tony has older twin brothers. The product of the ages of all three brothers is 32. If all of their ages are integers, compute the sum of their ages.

S11S4 Two cylindrical jars have the same volume. The diameter of the second jar is 50% more than the diameter of the first jar, and the height of the second jar is x% less than the height of the first jar. Compute x.

Part III   Spring 2011   Contest 1   Time: 10 Minutes

S11S5 Compute the number of real values of x so that $\sqrt{143 - \sqrt{x}}$ is an integer.

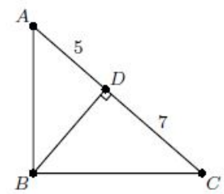
S11S6 Let S be the set of permutations (orderings) of the numbers $1, 2, 3, 4, 5, 6, 7$ for which the first term is neither 1 nor 2. A permutation is chosen randomly from S. Compute the probability that the second term of the permutation is 6.

New York City Interscholastic Math League
Soph-Frosh Division   Contest Number 2

Part I | Spring 2011 | Contest 2 | Time: 10 Minutes

S11S7 The two lines $x = \frac{3}{2}y + a$ and $y = \frac{3}{2}x + b$ intersect at the point $(1, 2)$. Compute the product ab.

S11S8 Right triangle ABC (with right angle at B) is shown below in the diagram, where $AD = 5$, $DC = 7$, and BD meets AC at a right angle. Compute the area of $\triangle ABC$.



Part II | Spring 2011 | Contest 2 | Time: 10 Minutes

S11S9 The numbers $2, 4, 6, 8, 10, 12, 14$, and 16 are placed at the vertices of a cube (one number per vertex) so that the sum of the four numbers on each face of the cube is the same. Compute this sum.

S11S10 Suppose that $4^a = 7$, $7^b = 11$, and $11^c = 16$. Compute abc.

Part III | Spring 2011 | Contest 2 | Time: 10 Minutes

S11S11 Compute the product

$$(\sqrt{7} + \sqrt{10} + \sqrt{11})(-\sqrt{7} + \sqrt{10} + \sqrt{11})(14 - 2\sqrt{110})$$

S11S12 Compute both values of k for which the following equation has no real solution in x:

$$\frac{x-7}{x-3} = \frac{x-k}{x-6}$$

New York City Interscholastic Math League
Soph-Frosh Division Contest Number 3

Part I Spring 2011 Contest 3 Time: 10 Minutes

S11S13 Compute the number of points at which the graph of $f(x) = 4x^2 + 12x + 9$ intersects the x axis.

S11S14 The average of the numbers $1, 2, 3, ..., 99, 100, x$, and y is 100. Compute $x + y$.

Part II Spring 2011 Contest 3 Time: 10 Minutes

S11S15 A square of area 64 is inscribed in a circle (that is, each vertex of the square is on the circle). Compute the area of the circle.

S11S16 There are seven plain white cards. Of those seven cards, four of them are marked A and the remaining three are marked B. If cards are randomly arranged in a straight line, compute the probability that the arrangement is $AABBABA$.

Part III Spring 2011 Contest 3 Time: 10 Minutes

S11S17 Compute the sum of all the roots of $(x+3)(2x+3)+(3x+1)(x+3)+(2x+6)(x+5) = 0$.

S11S18 Real numbers x, y, z satisfy the following equations:

$$x + \frac{1}{y} = 7, \quad y + \frac{1}{z} = 1, \quad z + \frac{1}{x} = \frac{5}{3}$$

Compute xyz.

New York City Interscholastic Math League
Soph-Frosh Division — Contest Number 1 Solutions

S11S1. **3**. The solution only exists if $x - 2.5 \geq 0$ AND $x - 3.5 \leq 0$. (It is impossible for $x - 2.5 < 0$ AND $x - 3.5 > 0$, and if both $x - 2.5$ and $x - 3.5$ have the same sign then $x - 2.5 = x - 3.5$, which is impossible). Then $|x - 2.5| = x - 2.5$ and $|x - 3.5| = -(x - 3.5)$.

$$x - 2.5 = -x + 3.5 \implies 2x = 6$$

$$x = 3$$

Alternate Solution. Recall that $|a - b|$ is the distance between a and b on the real line. The answer is thus the number equidistant from 2.5 and 3.5, namely 3.

S11S2. $\mathbf{\frac{3}{8}}$. $|ab - cd|$ is odd if and only if the parity of ab and cd are opposite (i.e. one is odd while the other is even). This can occur if ab is odd and cd is even, or if ab is even and cd is odd. The probability that ab is odd is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, and the probability that cd is even is $1 - \frac{1}{4} = \frac{3}{4}$. Thus, the total probability for the first case is $\frac{3}{16}$. Similarly, the total probability for the second case is $\frac{3}{16}$, for a combined probability of $\frac{3}{8}$.

S11S3. **10**. If Tony's age is x and his twin brothers are y, then we have $x \cdot y^2 = 32 = 2^5$. Since x and y are integers, they must be powers of two. Since Tony is younger ($x < y$), this means that $x = 2$ and $y = 4$. Then, the sum of all three ages is $2 + 4 + 4 = 10$.

S11S4. $\mathbf{\frac{500}{9}}$. Let r be the radius of the first cylinder and h the height of the first cylinder. The volume of the cylinder is $V = \pi r^2 h$. The second cylinder has diameter increased by 50%, so its radius is $1.5r$. If the height of the new cylinder is h_{new}, we have that

$$V = \pi r^2 h = \pi (1.5r)^2 h_{\text{new}}$$

$$h = h_{\text{new}} \frac{9}{4} \implies h_{\text{new}} = \frac{4}{9} h$$

That is, the height is decreased by $\frac{5}{9} \cdot 100\%$, so $x = \frac{500}{9}$.

S11S5. **12**. Since $\sqrt{x}$ is defined to be nonnegative, we know that

$$\sqrt{143 - \sqrt{x}} \leq \sqrt{143}$$

Since

$$\sqrt{143 - \sqrt{x}} \geq 0$$

this tells us that it must be at least zero, but less than 12, so the only possible values for $\sqrt{143 - \sqrt{x}}$ are between 0 and 11, inclusive. All of these values can be achieved: if

$\sqrt{143 - \sqrt{x}} = m$ for m between 0 and 11, then we let $x = (m^2 - 143)^2$, and we can verify that all numbers inside the square roots are nonnegative for these cases.

S11S6. $\mathbf{\frac{2}{15}}$. The probability that a randomly selected permutation satisfies that property is the number of permutations with that property, divided by the number of permutations in S. We can count the number of permutations in S in two ways. First, we could note that there are five choices for the first element (all but 1 or 2), and then everything else (six choices) for the second element, then five for the third, and so on, for a total of $5 \cdot 6!$ permutations. Or we could note that there are 7! permutations of $1, 2, 3, 4, 5, 6, 7$ and that we are excluding the 6! that start with one and the 6! that start with two, for a total of $7! - 2(6!) = (7 - 2)6! = 5 \cdot 6!$. Now we count the number of permutations with the second element 6. There are four choices for the first term (3, 4, 5, or 7). The second term is fixed. Then, there are five choices for the third term, four for the fourth term, and so on. Thus, there are $4 \cdot 5!$ permutations that satisfy the property. The probability of a randomly chosen permutation satisfying the property is

$$\frac{4 \cdot 5!}{5 \cdot 6!} = \frac{4 \cdot 5!}{5 \cdot 6 \cdot 5!} = \frac{4}{30} = \frac{2}{15}$$

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S11S7. $-\mathbf{1}$. Substituting $x = 1$ and $y = 2$ into both linear equations, we get that $a = -2$ and $b = \frac{1}{2}$. Thus their product is -1.

S11S8. $\mathbf{6\sqrt{35}}$. Let $BD = h$. Then, by the Pythagorean Theorem, AB and BC can be expressed as

$$AB = \sqrt{25 + h^2}$$
$$BC = \sqrt{49 + h^2}$$

The Pythagorean Theorem on $\triangle ABC$ tells us that

$$AB^2 + BC^2 = AC^2$$

$$(25 + h^2) + (49 + h^2) = 12^2$$

This tells us that $2h^2 = 70$ or

$$h = \sqrt{35}$$

Then, the area of $\triangle ABC$ is

$$\frac{(AC)(h)}{2} = 6\sqrt{35}$$

Alternate Solution By the angle-angle triangle similarity theorem, triangles ADB and BDC are similar, with similarity ratio a. Then, we know that

$$\frac{DC}{DB} = \frac{DB}{AD}$$

Since $DC = 7$ and $AD = 5$, this tells us that $BD = \sqrt{35}$. As above, the area is $6\sqrt{35}$.

S11S9. **36**. There is no need to arrange the numbers to compute the sum! The sum of the numbers on the top face, t, is equal to the sum of the numbers on the bottom face, b. We know that $t + b$ is the sum of $2, ..., 16 = 72$, and since $t = b$ this is 36. (To show that this is possible, we use the arrangement where the front face is 10, 16, 4, 6 clockwise from the top left, and the back face is 8, 2, 16, 12 again clockwise from the top left).

CHALLENGE: Show that there are, up to reflections and rotations, exactly three ways to put these numbers on the vertices of a cube so that the sum of numbers on each face is equal to 18.

S11S10. **2**. Raising the first equation, $4^a = 7$, to the power b, we get that

$$11 = 7^b = (4^a)^b = 4^{ab}$$

Then, we raise this to the power c:

$$16 = 11^c = (4^{ab})^c = 4^{abc}$$

which means that $abc = 2$.

S11S11. $\mathbf{-244}$. Repeatedly using the difference of squares factoring ($(a+b)(a-b) = a^2-b^2$), the given expression can be simplified to

$$\begin{aligned}(\sqrt{7}+\sqrt{10}+\sqrt{11})(-\sqrt{7}+\sqrt{10}+\sqrt{11})(14-2\sqrt{110}) &= ((\sqrt{10}+\sqrt{11})^2-(\sqrt{7})^2)(14-2\sqrt{110}) \\ &= (14+2\sqrt{110})(14-2\sqrt{110}) \\ &= 196-(2\sqrt{110})^2 \\ &= 196-440 = -244\end{aligned}$$

S11S12. **10 and 6**. Our plan is to solve for x as a function of k, but we note that if we get $x = 3$ or $x = 6$ as the resulting solution, we have created an invalid solution because the denominators are zero. We cross-multiply to get

$$(x-6)(x-7) = (x-3)(x-k)$$

and then expand to get

$$x^2 - 13x + 42 = x^2 - (k+3)x + 3k$$

That is,

$$(k-10)x = 3k - 42$$

or

$$x = \frac{3k-42}{k-10} = 3 - \frac{12}{k-10}$$

There is no real solution for x when (and only when) $k = 10$. Additionally, when we have $k = 6$, we get $x = 6$, which is an invalid solution. There is no value of k that yields the invalid solution $x = 3$, so the only values of k that do not yield a valid real solution for x are 10 and 6.

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S11S13. **1**. The graph intersects the x-axis when $f(x) = 0$. Because

$$f(x) = 4x^2 + 12x + 9 = (2x + 3)^2$$

the graph only intersects the x-axis only once, at $x = \frac{-3}{2}$.

S11S14. **5150**. The arithmetic mean of the set is the sum divided by the size of the set:

$$\frac{1 + 2 + 3 + ... + 99 + 100 + x + y}{102} = 100$$

We know the sum $1+2+3+...+99+100$ by the general formula for the sum of an arithmetic sequence:

$$1 + 2 + 3 + ... + 99 + 100 = \frac{101 \cdot 100}{2} = 5050$$

Then, we have

$$\frac{5050 + x + y}{102} = 100$$

which tells us that

$$x + y = 5150$$

S11S15. **32π**. Since the area of a square is s^2, the side length of the square is 8. Then we use the Pythagorean Theorem:

$$r^2 + r^2 = 8^2 \implies r = 4\sqrt{2}$$

Hence, the area of the circle is $\pi r^2 = 32\pi$.

Alternate Solution. We again use the fact that the side length of the square is 8. Then, the diameter of the circle satisfies

$$8^2 + 8^2 = d^2 \implies d = 8\sqrt{2}$$

The radius is half the diameter, or $4\sqrt{2}$. Thus, the area is 32π.

S11S16. $\mathbf{\frac{1}{35}}$. There are 7! total arrangements of the seven cards. There are $4! \cdot 3!$ arrangements satisfying the requested property: 4! ways to arrange the As, and 3! ways to arrange the Bs. Thus the answer is

$$\frac{3! \cdot 4!}{7!} = \frac{6}{5 \cdot 6 \cdot 7} = \frac{1}{35}$$

Alternate Solution. There are $\binom{7}{3}$ strings consisting of four A's and three B's, so the requested probability is $\frac{1}{\binom{7}{3}} = \frac{1}{35}$.

S11S17. $\mathbf{-5}$. The given equation can be factored:

$$(x+3)(7x+14) = 0$$

This has roots at $x = -3$ and $x = -2$, the sum of which is -5.

S11S18. $\mathbf{1}$. If we add all three equations, we get:

$$x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{29}{3}.$$

If we multiply all three equations, we get:

$$xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz} = \frac{35}{3}.$$

Note that the middle six terms add to $\frac{29}{3}$. Hence

$$\frac{29}{3} + \frac{1}{xyz} = \frac{35}{3}.$$

Therefore

$$xyz + \frac{1}{xyz} = 2$$

so

$$(xyz)^2 - 2(xyz) + 1 = 0$$

Thus

$$xyz = 1.$$

We can construct a solution: we have that $x = \frac{1}{yz}$. We have that $\frac{1}{yz} + \frac{1}{y} = 7$, which means that $1 + z = 7yz$. The second equation tells us that $yz + 1 = z$, or $7yz = 7z - 7$. Then, $1 + z = 7z - 7$, which means that $z = \frac{4}{3}$. The second equation tells us that $\frac{4}{3}y + 1 = \frac{4}{3}$, which means that $y = \frac{1}{4}$, and $x = 3$. We verify that this satisfies all of the given equations.