New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION | **CONTEST NUMBER TWO** | **SPRING 2007**

PART I: 10 minutes | **NYCIML Contest Two** | **Spring 2007**

S07B07 The point $(k,0)$ is equidistant from the origin and the point $(2,6)$. Compute the value of k.

S07B08 If $\sqrt{x\sqrt[3]{x\sqrt[4]{x}}} = x^h$, compute the value of h.

PART II: 10 minutes | **NYCIML Contest Two** | **Spring 2007**

S07B09 A circle is inscribed in rhombus with diagonals of length 6 and 8. Compute the radius of the circle.

S07B10 Compute x, if $\log_4(x+2) = \log_2(2x-1)$.

PART III: 10 minutes | **NYCIML Contest Two** | **Spring 2007**

S07B11 In ΔABC, point D lies on side $\overline{AC}$ such that $AD : DC = 3 : 2$. Point E lies on side $\overline{BC}$ such that $CE : EB = 1 : 5$. If the area of ΔABC is 75, compute the area of ΔCDE.

S07B12 The sum of the terms of an infinite geometric progression is $\frac{1}{2}$ and the sum of its first three terms is $\frac{1}{6}$. Compute the common ratio of the geometric progression.

ANSWERS

7. **10**
8. $\mathbf{\frac{17}{24}}$
9. $\mathbf{\frac{12}{5}}$
10. $\mathbf{\frac{5+\sqrt{41}}{8}}$
11. **5**
12. $\mathbf{\frac{\sqrt[3]{18}}{3}}$

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SPRING 2007 Solutions

7. Using the distance formula, we know that $k=\sqrt{(k-2)^2+(6-0)^2}$ or $k^2=k^2-4k+4+36$. Solving for k yields $k=10$.

8. We can begin by rewriting $\sqrt{x\cdot\sqrt[3]{x\cdot\sqrt[4]{x}}}=x^h$ as $\{x[x(x^{1/4})]^{1/3}\}^{1/2}=x^h$ and so $\{x[x^{5/4}]^{1/3}\}^{1/2}=\{x[x^{5/12}]\}^{1/2}=\{x^{17/12}\}^{1/2}=x^{17/24}$. Therefore $h=\frac{17}{24}$.

9. Since the diagonals of a rhombus are perpendicular bisectors of one another, we can compute the side of the rhombus to be 5. The area of the rhombus is now given by $K=\frac{1}{2}d_1d_2$ where d_1 and d_2 denote the diagonals and thus $K=\frac{1}{2}(6\cdot 8)=24$. But the area of any quadrilateral circumscribed around a circle is given by $K=rs$ where r denotes the radius of the inscribed circle and s the semiperimeter. Therefore $24=r\cdot\left(\frac{4\cdot 5}{2}\right)$ and $r=\frac{12}{5}$.

10. Let the common value of the two logarithms be called M. Then the equation may be rewritten as $4^M=x+2$, $2^M=2x-1$. Since 4^M is the square of 2^M, we see that $x+2=(2x-1)^2=4x^2-4x+1$. The roots of this quadratic equation are $\frac{1}{8}(5\pm\sqrt{41})$. Choosing the negative sign yields an x for which $2x-1<0$, producing an undefined logarithm. Thus the unique solution is $x=\frac{1}{8}(5+\sqrt{41})$.

11. The area of ΔCDE is given by $\frac{1}{2}(CD)(CE)\sin(\angle ACB)$. The area of $\Delta ABC=75=\frac{1}{2}(AC)(BC)\sin(\angle ACB)$. Thus $CD=\frac{2}{5}(AC)$ and $CE=\frac{1}{6}(BC)$. Therefore, the area of ΔCDE is $\left(\frac{2}{5}\right)\left(\frac{1}{6}\right)(75)=5$.

12. Let the progression be written $(a, ar, ar^2, ar^3, \ldots)$, where the common ratio r satisfies $-1<r<1$. The sum of the series is $a/(1-r)$. The sum of the first n terms is $a\left(1-r^n\right)/(1-r)$. With the information given, these formulas lead to the equations $2a=1-r$ and $6a\left(1-r^3\right)=1-r$. It follows that $3\left(1-r^3\right)=1$, and therefore $r=\sqrt[3]{\frac{2}{3}}=\frac{\sqrt[3]{18}}{3}$.

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SENIOR B DIVISION | **CONTEST NUMBER THREE** | **SPRING 2007**

PART I: 10 minutes | **NYCIML Contest Three** | **Spring 2007**

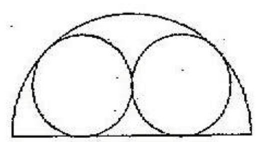
S07B13 A cube with edge of length 10 is painted purple and is then cut into 1000 unit cubes. Compute the number of these unit cubes that have at least one face painted.

S07B14 The sum of the first 5 terms of an arithmetic progression is equal to the sum of the 6^{th} and 7^{th} terms. If the fourth term is 70, compute the value of the second term.

PART II: 10 minutes | **NYCIML Contest Three** | **Spring 2007**

S07B15 If $f(x)=ax+b$, and $f(f(f(x)))=27x+39$, compute (a,b).

S07B16 Two externally tangent circles of radius 5 are inscribed in a semicircle of radius r. Compute r.



PART III: 10 minutes | **NYCIML Contest Three** | **Spring 2007**

S07B17 Compute the value of
$(\tan 10^\circ)(\tan 20^\circ)(\tan 30^\circ)(\tan 40^\circ)(\tan 50^\circ)(\tan 60^\circ)(\tan 70^\circ)(\tan 80^\circ)$

S07B18 Define a *snaking number* to be a positive integer whose decimal representation is of the form $(ABCD)_{10}$, where $A \neq 0$ and $A < C < B < D$. Compute the total number of snaking numbers.

ANSWERS

13. **488**
14. **28**
15. **(3, 3)**
16. $5\sqrt{2}+5$
17. **1**
18. **126**

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13. The unit cubes that have no faces painted are those in the interior of the larger cube. This interior cube has an edge of 8, and thus $8^3 = 512$ cubes that have no faces painted. Therefore, $1000 - 512 = 488$ cubes have at least one face painted.

14. Let $(a, a+d, a+2d, \ldots)$ denote the terms of the arithmetic sequence. Then, according to the given, $a + (a+d) + (a+2d) + (a+3d) + (a+4d) = (a+5d) + (a+6d)$ which upon simplification leads to $3a = d$. Since the fourth term of the sequence equals 70, we know that $a + 3d = 70$. Substituting $3a$ for d we have $a + 9a = 70$, or $a = 7$. Thus, $d = 3a = 21$ and the second term is $a + d = 28$.

15. Since $f(f(f(x))) = f(f(ax+b) = f(a(ax+b)+b) = f(a^2x+ab+b)$ $= a(a^2x+ab+b)+b = a^3x+a^2b+ab+b$ we have that $27x+39 = a^3x+a^2b+ab+b$ for all real x. So $a^3 = 27$ and $a^2b+ab+b = 39$, and $a = 3$ and $b = 3$.

16. Let the center of the semicircle be O, and let the two circles touch at P. By symmetry, the ray $\overline{OP}$ determines two quarter-circles, each circumscribed about one circle of radius 5. Choose one of these quarter-circles and draw the radius $\overline{OM}$ through the center A of its inscribed circle. Note that $\overline{AM}$ is a radius of the inscribed circle. Therefore $r = OM = OA + AM = 5\sqrt{2} + 5$.

17. For any acute angle θ, $\tan(90° - \theta) = \cot(\theta) = \dfrac{1}{\tan(\theta)}$. Thus, $\tan(90° - \theta) \cdot \tan(\theta) = 1$. Therefore, the product equals 1.

18. Since the four digits are distinct with A being smallest, all four digits must be positive. Now any selection of four digits from the set $\{1,2,3,4,5,6,7,8,9\}$ will produce a unique snaking number. For instance, $\{2,4,6,7\}$ yields 2647. It follows that the number of snaking numbers is ${}_9C_4 = 126$.

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SENIOR B DIVISION **CONTEST NUMBER FOUR** **SPRING 2007**

PART I: 10 minutes **NYCIML Contest Four** **Spring 2007**

S07B19 The arithmetic mean of two positive numbers is 7. Their geometric mean is 4. Compute the sum of their squares.

S07B20 Compute all values of k such that the equation $x^2 + kx + 2k = 0$ has exactly one real solution.

PART II: 10 minutes **NYCIML Contest Four** **Spring 2007**

S07B21 If $\sqrt{8+\sqrt{15}} - \sqrt{8-\sqrt{15}} = \sqrt{j}$ for integer j, compute the value of j.

S07B22 Compute all values of x such that $|2x| + |x+2| = 3$.

PART III: 10 minutes **NYCIML Contest Four** **Spring 2007**

S07B23 Compute the sum of the digits of 11^6.

S07B24 Let R be the length of the radius of the circumscribed circle, and let r be the length of the radius of the inscribed circle of a triangle with sides of length 8, 8 and 2. Compute R/r.

ANSWERS

19.	**164**	22.	$\mathbf{-1, \frac{1}{3}}$
20.	**0, 8**	23.	**28**
21.	**2**	24.	$\mathbf{\frac{32}{7}}$

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Senior B Division CONTEST NUMBER 4

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19. If x and y are our numbers, then we know that $x + y = 14$ and $xy = 16$. Squaring the first expression yields $(x+y)^2 = x^2 + 2xy + y^2 = 196$. Thus, $x^2 + y^2 = 196 - 2xy = 196 - 32 = 164$.

20. If $x^2 + kx + 2k = 0$ has exactly one real root r, then r must be a double root. In using quadratic formula to solve for x, the discriminant must equal zero. This yields $(k)^2 - 4(1)(2k) = k^2 - 8k = 0$. Thus, $k = 0$ or $k = 8$.

21. Squaring both sides of the equation we have $\left(8+\sqrt{15}\right) + \left(8-\sqrt{15}\right) - 2\sqrt{64-15} = j$, $j = 2$.

22. Note that if $2x \geq 0$ then also $x + 2 > 0$. In this case both absolute value signs may be deleted, and we have $2x + x + 2 = 3$, or $x = \frac{1}{3}$. The other alternative is $2x \leq 0$. In this case we must solve the two equations $-2x \pm (x+2) = 3$. This leads to $x = -1$ and $x = -\frac{5}{3}$. However, only $x = -1$ checks in the original equation. Solutions: $\{-1, \frac{1}{3}\}$.

23. One way, of course, is to multiply this out. However, another way is to think of this as the binomial expansion of $(10+1)^6$. We can look at the 6th row of Pascal's triangle to compute this: 1 6 15 20 15 6 1 yields the number 1,771,561. The sum of the digits is thus 28. Or we could have just added the digits of the 6th row!

24. Let the vertex angle appear at A, and the base angles at B and C. Let F be the midpoint of BC. By symmetry, the circumcenter O falls on the line AF. Note that $AF = \sqrt{8^2 - 1^2} = \sqrt{63} = 3\sqrt{7}$. Now ΔBOF is a right triangle with hypotenuse $OB = R$ and legs $BF = 1$ and $OF = AF - AO = 3\sqrt{7} - R$. Using the Pythagorean Theorem once more, $R^2 = 1 + (3\sqrt{7} - R)^2$, so that $R = 32/3\sqrt{7}$. We can compute r from the formula $K = rs$, in which K is the area of the triangle, and s is the semiperimeter. This gives $r = \frac{1}{3}\sqrt{7}$. Finally, $R/r = 32/7$.

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SENIOR B DIVISION	**CONTEST NUMBER FIVE**	**SPRING 2007**
PART I: 10 minutes	**NYCIML Contest Five**	**Spring 2007**

S07B25 One half of two more than a positive number is twice the reciprocal of the positive number. Compute the number.

S07B26 Nine distinct lines lie in a plane. Compute the maximum number of points of intersection.

PART II: 10 minutes	**NYCIML Contest Five**	**Spring 2007**

S07B27 Compute the value of $\sin\left(2Arc\cos\frac{5}{13}\right)$.

S07B28 Each of the numbers 228, 344, and 518 leave the same remainder when divided by the positive integer x. Compute the maximum possible value of x.

PART III: 10 minutes	**NYCIML Contest Five**	**Spring 2007**

S07B29 Compute all values of x such that $\sqrt{x-6} = x\sqrt{x-6}$.

S07B30 Compute the maximum real value of y for which $2x^2 - \sqrt{3}xy + y^2 = 1$ has at least one real solution for x.

ANSWERS

25.	$-1+\sqrt{5}$	28.	**58**
26.	**36**	29.	**1, 6**
27.	$\frac{120}{169}$	30.	$\frac{2\sqrt{10}}{5}$

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25. The statement gives us the following proportion: $\frac{x+2}{2}=\frac{2}{x}$. This yields the quadratic equation $x^2+2x-4=0$. Solving for the positive value of x gives us $x=\frac{-2+2\sqrt{5}}{2}=-1+\sqrt{5}$.

26. Since we are looking for the maximum number of points of intersection, we want the lines to be such that no three are concurrent and no two are parallel. The number of intersection points then becomes ${}_9C_2=\frac{9!}{2!7!}=\frac{9\times 8}{2\times 1}=36$.

27. Let $\text{Arccos}\left(\frac{5}{13}\right)=\theta$. It then follows that $\cos\theta=\frac{5}{13}$ and $\sin\theta=\frac{12}{13}$. Thus $\sin\left(2\cdot\text{Arccos}\left(\frac{5}{13}\right)\right)=\sin(2\theta)=2\sin\theta\cos\theta=2\cdot\frac{12}{13}\cdot\frac{5}{13}=\frac{120}{169}$.

28. The given condition may be written as $228\equiv 344\equiv 518 \pmod{x}$. Subtracting 228 from each member yields the equivalent condition $0\equiv 116\equiv 290 \pmod{x}$. Thus x is a common divisor of 116 and 290. If x is maximal, then $x=\gcd(116,290)=58$.

29. If $x=6$, then both sides are $=0$, if $x\neq 6\rightarrow x=\frac{\sqrt{x-6}}{\sqrt{x-6}}=1$.

30. Treating the equation as a quadratic in y, and choosing the plus sign to enable a maximum value, we get $y=\frac{1}{2}\left\{\sqrt{3}x+\sqrt{4-5x^2}\right\}$. We will use an important inequality that deals with sums of square roots: $\sqrt{px}+\sqrt{qy}\leq\sqrt{(p+q)(x+y)}$, with equality if and only if $py=qx$. This can be proven by squaring both sides, subtracting identical terms from both sides, and applying the AM-GM inequality to what remains. In our case, we see that $2y=\sqrt{3x^2}+\sqrt{4-5x^2}\leq\sqrt{(3+5)(x^2+(\frac{4}{5}-x^2))}=\frac{4}{5}\sqrt{10}$. Thus an upper bound for y is $\frac{2}{5}\sqrt{10}$. Moreover, this bound is attainable, namely when $3(\frac{4}{5}-x^2)=5x^2$, or $x=\frac{\sqrt{30}}{10}$. Hence the answer is $\frac{2}{5}\sqrt{10}$.

In the alternative, if we let $y=k$ be the maximum value of y, then we obtain $2x^2-\sqrt{3}xk+k^2-1=0$. Notice that this is a quadratic in x. Solving for x yields

$$x=\frac{\sqrt{3}k\pm\sqrt{3k^2-4(2)(k^2-1)}}{4}=\frac{\sqrt{3}k\pm\sqrt{8-5k^2}}{4}.$$

The maximum value of k occurs when $8-5k^2=0\rightarrow k=\frac{2\sqrt{10}}{5}$. Note: this equation is actually the graph of an ellipse.

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SENIOR B DIVISION **CONTEST NUMBER ONE** **SPRING 2007**

PART I: 10 minutes **NYCIML Contest One** **Spring 2007**

S07B01 The mean of five numbers is 76, the median is 75, the mode is 81, and the range is 11. Compute the value of the second smallest number.

S07B02 Compute the number of arrangements of the letters in the word *COCOROS* that have three consecutive *O*'s.

PART II: 10 minutes **NYCIML Contest One** **Spring 2007**

S07B03 Compute the number of positive integers less than 2007 that have an odd number of positive integer factors.

S07B04 The roots of $x^2 + ax + b = 0$ are additive inverses of each other and also multiplicative inverses of each other. Compute both roots.

PART III: 10 minutes **NYCIML Contest One** **Spring 2007**

S07B05 Compute the number of points of intersection of the graphs of the equations $x^2 - y^2 = 0$ and $2x^2 + 3y^2 = 1$.

S07B06 In ΔABC, medians $\overline{AD}$ and $\overline{BE}$ intersect at a point F. If ΔABF is equilateral and $AB = 10$, compute the perimeter of ΔABC.

ANSWERS

1.	**73**	3.	**44**	5.	**4**
2.	**60**	4.	$\pm i$	6.	$10 + 20\sqrt{7}$

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Senior B Division CONTEST NUMBER 1

SPRING 2007 Solutions

1. If we call the numbers *a, b, c, d* and *e* such that $a \le b \le c \le d \le e$, we know that $c = 75$. Moreover, $d = e = 81$. This leads to $a = 70$. The sum of the numbers is $5 \times 76 = 380$. Thus $b = 73$.

2. Imagine that the three *O*'s were combined to form a new letter O_1. The number of arrangements of the letters *C, C, R, S* and O_1 is now $\frac{5!}{2!} = 60$.

3. Any integer having an odd number of positive divisors is a perfect square. Thus, we are looking for the number of perfect squares less than 2007, which is given by $\lfloor \sqrt{2007} \rfloor = 44$.

4. The sum of the roots is $-a$, but is also 0, since they are additive inverses. Thus $a = 0$. The product of the roots is b, but is also 1, since they are multiplicative inverses. Thus $b = 1$. Our quadratic is now just $x^2 + 1 = 0$, and the roots are $\pm i$.

5. Knowing that $x^2 = y^2$ allows us to substitute and obtain $5x^2 = 1 \rightarrow x = \pm\frac{\sqrt{5}}{5}$. Each value of *x* in turn gives us two values of *y*. Therefore, there are **four** points of intersection, namely $\left(\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right); \left(\frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right); \left(-\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$ and $\left(-\frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right)$. Incidentally, the first equation is two intersecting lines (a degenerate hyperbola), and the second equation is an ellipse.

6. $AF = BF = AB = 10$. Since any two medians trisect each other, we also have $AD = \frac{3}{2}AF = 15 = \frac{3}{2}BF = BE$. Let $x = AE$. Using Stewart's Theorem on ΔABE, we get $(5)(10)(15) + (10)(15)(10) = (10)(5)(10) + 10x^2$. Thus $x^2 = 175$, and we find that $x = 5\sqrt{7}$. By symmetry, $AE = EC = CD = DB = x$. Hence the perimeter of ΔABC is $10 + 4x = 10 + 20\sqrt{7}$.

In the alternative, we can establish that $\angle DFB = 120°$, and we can use the law of cosines to obtain $BD = \sqrt{(10)^2 + (5)^2 - 2(10)(5)\cos(120°)} = \sqrt{175} = 5\sqrt{7}$.