NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 1

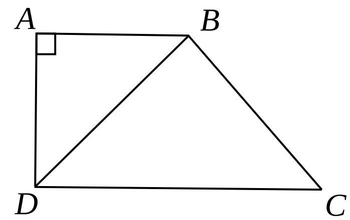
PART I   ***FALL 2010***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F10B01 The circumference of a circle is $a\pi$ centimeters and the area of the same circle is $4a\pi$ square centimeters, where $a > 0$. Compute the numerical value of a.

F10B02 The European research department at the United Nations employs 30 people, all of whom speak French, Italian, or Spanish. Five of them speak all three languages. Nine speak French and Spanish. Twenty speak Italian, of which 12 also speak French. Eighteen speak Spanish. No one speaks only French. Compute how many speak only Spanish.

PART II   ***FALL 2010***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F10B03 The diagram at right shows trapezoid $ABCD$ with $\overline{AB} \perp \overline{AD}$, $\overline{CD} \perp \overline{AD}$, $\angle ABD = 45°$, and $CD = AB + 6$. If the area of the trapezoid is 88 square inches, compute the length of $\overline{AB}$ in inches.



F10B04 For $x > 0$, compute all ordered triples of ***integers*** (x, y, z) that satisfy the following system of equations:

$$x + y + z = 9$$
$$z^x = y^{2x}$$
$$3^z = 3 \cdot 3^x$$

PART III   ***FALL 2010***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F10B05 Write the sum $\sqrt{\frac{7}{4}+\sqrt{3}}+\sqrt{\frac{7}{4}-\sqrt{3}}$ as an integer in simplest form.

F10B06 If $x^3 + y^3 = 1$ and $x + y = \frac{3}{2}$, compute xy.

ANSWERS TO CONTEST 1

F10B01. 16

F10B02. 6

F10B03. 8

F10B04. (3, 2, 4)

F10B05. 2

F10B06. $\frac{19}{36}$

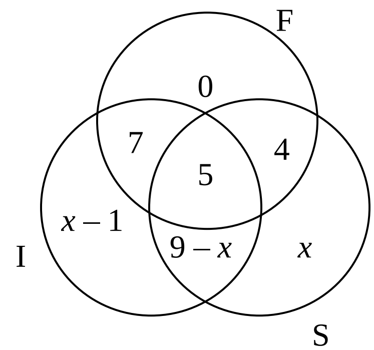
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 1

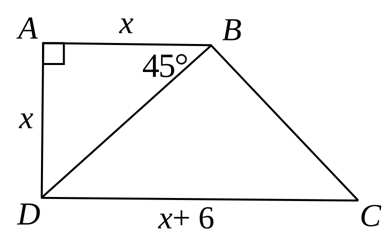
Fall 2010 Solutions

F10B01 **16**. Let r = the radius of the circle. Since the circumference of the circle is $a\pi$, we have $a = 2r \Rightarrow r = \frac{a}{2} \Rightarrow r^2 = \frac{a^2}{4}$. Also, since the area of the circle is $4a\pi$, we have $r^2 = 4a$. Using substitution, $\frac{a^2}{4} = 4a \Rightarrow a = 16$.

F10B02 **6**. In the Venn diagram, the overlapping circles represent the sets of people speaking French (F), Italian (I), and Spanish (S). If 5 peoplespeak all three languages and 9 speak both French and Spanish, then 4 speak French and Spanish but not Italian. Similarly, if 12 people speak both French and Italian, then 7 of them do not also speak Spanish. Now, let x = the number of people speaking only Spanish. If there are 9 people who speak both French and Spanish and a total of 18 Spanish speakers, there must be $9 - x$ people who speak only Spanish and Italian. For there to be a total of 20 Italian speakers, the number of people who speak only Italian must be $20 - (7 + 5 + 9 - x) = 20 - 21 + x = x - 1$. Finally, the sum of all areas of the Venn diagram must equal 30, so we can solve for x: $0 + 7 + 5 + 4 + (x - 1) + (9 - x) + x = 30 \Rightarrow 24 + x = 30 \Rightarrow x = 6$. Alternately, we know that 20 people speak Italian, and 4 people speak Spanish and French but not Italian. Other than those 24 people, everyone must speak either only French or only Spanish. Since no one speaks only French, all six of the remaining people speak only Spanish.



F10B03 **8**. Let $x = AB$. Hence, $AD = x$ because $\angle ABD = 45°$ making ΔABD an isosceles right triangle. As a result, $CD = x + 6$. Now, using the formula for the area of a trapezoid $\left(\frac{1}{2}h(b_1 + b_2)\right)$, we have $\frac{1}{2}x(2x+6) = 88$ $\Rightarrow x^2 + 3x = 88 \Rightarrow x^2 + 3x - 88 = 0 \Rightarrow (x+11)(x-8) = 0 \Rightarrow x = 8$.



F10B04 **(3, 2, 4).** From the second equation it must be true either that $z = y^2$ or $z = -y^2$. From the third equation it must be true that $x = z - 1$. Combining these equations, we know that either $x = y^2 - 1$ or $x = -y^2 - 1$. Since $x > 0$, we can eliminate the second possibility, so we can now substitute $z = y^2$ and $x = y^2 - 1$ into the first equation, yielding $y^2 - 1 + y + y^2 = 9$. Simplifying and solving this last equation for y yields $2y^2 + y - 10 = 0 \Rightarrow (2y + 5)(y - 2) = 0 \Rightarrow y = -\frac{5}{2}$ or 2. Since y must be an integer, $y = 2$ and $(x, y, z) = (3, 2, 4)$.

F10B05 **2.** Let $N = \sqrt{\frac{7}{4} + \sqrt{3}} + \sqrt{\frac{7}{4} - \sqrt{3}}$ and note that $N > 0$. Square N and then simplify:

$$N^2 = \left(\frac{7}{4} + \sqrt{3}\right) + 2\sqrt{\left(\frac{7}{4} + \sqrt{3}\right)\left(\frac{7}{4} - \sqrt{3}\right)} + \left(\frac{7}{4} - \sqrt{3}\right) = \frac{14}{4} + 2\sqrt{\frac{49}{16} - 3} = \frac{14}{4} + 2\sqrt{\frac{1}{16}} = \frac{14}{4} + \frac{2}{4} = \frac{16}{4} = 4.$$

Therefore, $N^2 = 4$ and $N = 2$. Alternatively, it might be easy to "guess" the value of N by using approximations: $N \approx \sqrt{1.75 + 1.7} + \sqrt{1.75 - 1.7} \approx \sqrt{3.45} + \sqrt{0.05} \approx 2$.

F10B06 $\mathbf{\frac{19}{36}}$. Cube $x+y$, group and factor: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ $= (x^3 + y^3) + 3xy(x+y)$. Now substituting $\frac{3}{2}$ for $x+y$ and 1 for $x^3 + y^3$, we have $\left(\frac{3}{2}\right)^3 = 1 + 3xy\left(\frac{3}{2}\right)$ $\Rightarrow \frac{27}{8} = 1 + \frac{9xy}{2} \Rightarrow \frac{9xy}{2} = \frac{19}{8} \Rightarrow xy = \frac{38}{72} = \frac{19}{36}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 2

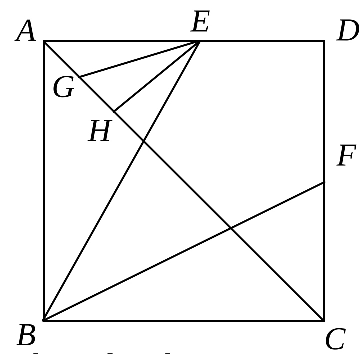
PART I ***FALL 2010*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F10B07 If the length of the edges of a given cube were increased by 1 inch, the volume would increase by 217 inches. Compute the length, in inches, of an edge of the cube.

F10B08 Homer and Ned agreed to play a game. If heads appeared on the toss of an ordinary coin, Ned had to double the amount of money that Homer had. If the result was tails, then Homer had to pay Ned $24. As it turned out, the coin came up heads, tails, heads, tails, heads, and then tails. After this series of tosses, Homer ended the game with exactly zero dollars (not owing Ned any money). Compute how much money Homer started with (in dollars).

PART II ***FALL 2010*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F10B09 The diagram at right shows square $ABCD$ with diagonal $\overline{AC}$. $\overline{BE}$ and $\overline{BF}$ trisect $\angle ABC$, and $\overline{EG}$ and $\overline{EH}$ trisect $\angle AEB$. Compute the ratio of the measure of $\angle EGH$ to the measure of $\angle EHG$.



F10B10 The nonzero three-digit base 9 number $\underline{A}\underline{B}\underline{C}_9$ is equal to the three-digit base 7 number $\underline{C}\underline{B}\underline{A}_7$. If A, B and C each represent a single digit, compute the value of this number in base 10.

PART III ***FALL 2010*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F10B11 The line $x+2y=8$ intersects the x- and y-axes at the points A and B respectively. Point C is on line segment $\overline{AB}$ such that $BC:CA=3:1$. A line drawn through C perpendicular to $\overline{AB}$ intersects the x-axis at the point $(t,0)$. Compute the value of t.

F10B12 Compute the sum, in degrees, of all of the values of x, where $0° \le x < 360°$, such that $\sin(60°+x)=2\sin x$.

ANSWERS TO CONTEST 2

F10B07. 8

F10B08. 21

F10B09. $\frac{13}{19}$

F10B10. 248

F10B11. $\frac{11}{2} = 5\frac{1}{2} = 5.5$

F10B12. $240°$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 2

Fall 2010 Solutions

F10B07 **8.** Let x = the length of one edge of the cube. We have $x^3 + 217 = (x + 1)^3$. Expanding the right hand side of the equation yields $x^3 + 217 = x^3 + 3x^2 + 3x + 1 \Rightarrow 3x^2 + 3x - 216 = 0 \Rightarrow x^2 + x - 72 = 0$. This quadratic equation can be solved by factoring: $x^2 + x - 72 = 0 \Rightarrow (x + 9)(x - 8) = 0$. After eliminating the negative root, the length of the edge of the cube must be 8 inches. Alternatively, search consecutive perfect cubes to find the pair whose difference is 217. Searching reveals $8^3 = 512$, $9^3 = 729$ and $9^3 - 8^3 = 729 - 512 = 217$, so the answer is 8.

F10B08 **21.** If Homer starts with x dollars, then after the first toss of heads he has $2x$. After tails comes up he has $2x - 24$. After heads he has $4x - 48$. After tails he has $4x - 72$. After heads he has $8x - 144$. After tails he has $8x - 168$ and he goes broke. So $8x - 168 = 0 \Rightarrow x = 21$. Alternatively, working backwards, Homer had \$24 before the last toss of tails. Then he had \$12 before heads, preceded by \$36 before tails. Then he had \$18 before heads. Then he had \$42 before tails and finally \$21 before the first toss of heads.

F10B09 $\mathbf{\frac{13}{19}}$. Since right $\angle ABC$ is trisected, $m\angle ABE = 30°$ and in right triangle ABC, $m\angle AEB = 60°$. As a result, since $\angle AEB$ is trisected, $m\angle AEG = m\angle GEH = m\angle HEB = 20°$. Also, $m\angle EAG = 45°$ because $\overline{AC}$ is a diagonal of the square. Now, $\angle EGH$ is an exterior angle to triangle AEG, and by the exterior angle theorem $m\angle EGH = 45° + 20° = 65°$. Next, using triangle AEH, $m\angle EHG = 180° - 45° - 40° = 95°$. Finally, the ratio is $\frac{m\angle EGH}{m\angle EHG} = \frac{65}{95} = \frac{13}{19}$.

F10B10 **248**. Notice that the digits A, B, and C are less than 7. Using the definitions of base 7 and base 9, we have $81A + 9B + C = 49C + 7B + A$. Solving this equation for B gives us $2B = 48C - 80A \Rightarrow B = 24C - 40A \Rightarrow B = 8(3C - 5A)$. Since A, B, and C are all integers and $B < 7$, it must be true that $B = 0$ and $3C - 5A = 0$. Therefore $C = 5$ and A = 3. Therefore the base 9 number is 305_9 and converting this to base 10 gives $305_9 = 3 \cdot 81 + 0 \cdot 9 + 5 = 248$.

F10B11 $\mathbf{\frac{11}{2} = 5\frac{1}{2} = 5.5}$. Point A is (8, 0) and point B is (0, 4). Therefore, since $BC : CA = 3 : 1$, point C is the coordinate pair (p, q) where $p = \frac{1 \cdot 0 + 3 \cdot 8}{3+1} = 6$ and $q = \frac{1 \cdot 4 + 3 \cdot 0}{3+1} = 1$. Next, the slope of $\overline{AB}$ is $-\frac{1}{2}$ implying that the slope of the line perpendicular to $\overline{AB}$ is 2. As a result, the point-slope equation of the line perpendicular to $\overline{AB}$ through point C is $y - 1 = 2(x - 6)$. Since this line passes through the x-axis at $(t, 0)$, by substitution we have $-1 = 2t - 12 \Rightarrow t = \frac{11}{2}$.

F10B12 **240º**. Using the angle addition formula for sines, we have $\sin(60° + x) = \sin 60° \cos x + \cos 60° \sin x = 2\sin x \Rightarrow \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = 2\sin x$. Multiplying both sides of the equation by 2 and collecting like terms gives us $\sqrt{3}\cos x + \sin x = 4\sin x \Rightarrow \sqrt{3}\cos x = 3\sin x$. Dividing both sides by $3\cos x$ yields $\frac{\sin x}{\cos x} = \frac{\sqrt{3}}{3} \Rightarrow \tan x = \frac{\sqrt{3}}{3}$ and as a result $x = 30°$ or $210°$. The sum of the solutions is $30° + 210° = 240°$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 3

PART I   ***FALL 2010***   ***CONTEST 3***   ***TIME: 10 MINUTES***

F10B13   Compute the sum of all positive integers n less than 10 for which $n! + (n + 1)!$ is divisible by 5.

F10B14   Compute the area of the triangle whose vertices are the center and y-intercepts of the circle $x^2 - 6x + y^2 + 12y + 20 = 0$.

PART II   ***FALL 2010***   ***CONTEST 3***   ***TIME: 10 MINUTES***

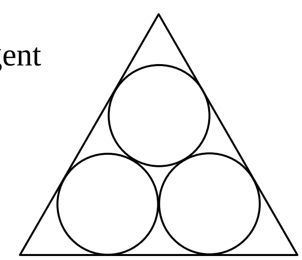
F10B15   Compute all real solutions of the equation $|x+2| = 2 \cdot |x-2|$.

F10B16   Compute the number of ordered pairs (X, Y), with $X < Y$, such that the three distinct natural numbers 21, X, and Y have a greatest common factor of 21 and a least common multiple of 462.

PART III   ***FALL 2010***   ***CONTEST 3***   ***TIME: 10 MINUTES***

F10B17   If $x + y + z = 6$, $xy + xz + yz= 11$, and $xyz = 6$, compute $\dfrac{x}{yz} + \dfrac{y}{xz} + \dfrac{z}{xy}$.

F10B18   The diagram at right shows three congruent circles, each tangent to the other two. The circles are circumscribed by an equilateral triangle. If the area of the triangle is $96 + 64\sqrt{3}$, compute the area of one of the circles.



ANSWERS TO CONTEST 3

F10B13. 38

F10B14. 12

F10B15. $\frac{2}{3}$, 6

F10B16. 3

F10B17. $\frac{7}{3} = 2\frac{1}{3} = 2.\overline{3}$

F10B18. 16π

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 3

Fall 2010 Solutions

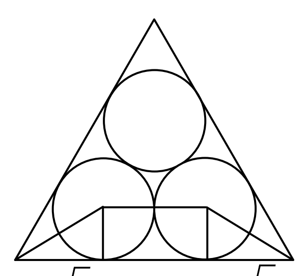
F10B13 **38**. Factor $n! + (n + 1)!$ into $n!(1 + n + 1) = n!(n + 2)$. This is divisible by 5 if $n \geq 5$ or $n + 2 = 5$, so it works for 3, 5, 6, 7, 8, and 9. The sum is 3 + 5 + 6 + 7 + 8 + 9 = 38.

F10B14 **12**. First, in order to put the equation of the circle into center-radius from, complete the square on x and y: $x^2 - 6x + 9 + y^2 + 12y + 36 = -20 + 9 + 36 \Rightarrow (x-3)^2 + (y+6)^2 = 25$. Hence, the center of the circle is $(3,-6)$. Next, the circle intersects the y-axis when $x = 0$. Using substitution we have $(0-3)^2 + (y+6)^2 = 25 \Rightarrow (y+6)^2 = 16 \Rightarrow y + 6 = \pm 4 \Rightarrow y = -2$, $y = -10$. Therefore, the base of the triangle is $-2-(-10) = 8$ and the height of the triangle is 3, so that its area is $\frac{1}{2} \cdot 8 \cdot 3 = 12$.

F10B15 $\mathbf{\frac{2}{3}}$ **and 6**. There are three cases to consider: i) $x \leq -2$, ii) $-2 < x < 2$, and iii) $x \geq 2$. In case i) we have $-(x+2) = -2(x-2) \Rightarrow x = 6$, which is disqualified because 6 is not in the given range. In case ii) we have $x + 2 = -2(x-2) \Rightarrow x + 2 = -2x + 4 \Rightarrow x = \frac{2}{3}$. Finally, in case iii) we have $x + 2 = 2(x-2) \Rightarrow x + 2 = 2x - 4 \Rightarrow x = 6$. Therefore, there are only two solutions: $x = \frac{2}{3}$ and 6.

F10B16 **3**. Observe that $\frac{462}{21} = 22$. Let $X = 21a$ and $Y = 21b$, where $a < b$ are distinct numbers such that the least common multiple of a and b is 22. Since 21, X, and Y must be distinct and $a \neq 1$, the possible pairs (a, b) are (2, 11), (2, 22) and (11, 22). Therefore, there are three possible pairs (X, Y) satisfying the conditions of the problem, namely (42, 231), (42, 462) and (231, 462).

F10B17 $\mathbf{\frac{7}{3} = 2\frac{1}{3} = 2.\bar{3}}$. First, observe that $\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} = \frac{x^2 + y^2 + z^2}{xyz}$. Now, observe that $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$, so that $x^2 + y^2 + z^2 = (x+y+z)^2 - 2(xy + xz + yz)$. Hence, we have $\frac{x^2 + y^2 + z^2}{xyz} = \frac{(x+y+z)^2 - 2(xy+xz+yz)}{xyz} = \frac{6^2 - 2(11)}{6} = \frac{18-11}{3} = \frac{7}{3}$. Alternatively, guess that the ordered triple $(3,2,1)$ satisfies the given information, so that $\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} = \frac{3}{2} + \frac{2}{3} + \frac{1}{6}$ $= \frac{9+4+1}{6} = \frac{14}{6} = \frac{7}{3}$.



F10B18 **16π**. The area of an equilateral triangle, given side length s, is $\frac{s^2}{4}\sqrt{3}$. Thus, $\frac{s^2}{4}\sqrt{3} = 96 + 64\sqrt{3} \Rightarrow s^2 = \frac{384}{\sqrt{3}} + 256 \Rightarrow s^2 = 128\sqrt{3} + 256$.Using 30-60-90 triangles, the sides of the triangle can be written in terms of the radiusr of one of the circles. Hence, $\left(2r + 2r\sqrt{3}\right)^2 = 256 + 128\sqrt{3} \Rightarrow 4r^2 + 8r^2\sqrt{3} + 12r^2 = 256 + 128\sqrt{3}$

$\Rightarrow 16r^2 = 256 \Rightarrow r^2 = 16$, and as a result, the area of the circle is 16π. Alternatively, we can see that $8r^2\sqrt{3} = 128\sqrt{3} \Rightarrow r^2 = 16$ in order to reach the same conclusion.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 4

PART I ***FALL 2010*** ***CONTEST 4*** ***TIME: 10 MINUTES***

F10B19 The points (0, 2), (1, 3) and (2, 2010) all lie on the graph of the parabola $y = ax^2 + bx + c$. Compute the ordered triple (a, b, c).

F10B20 A square and an equilateral triangle have equal perimeters. Circle *A* is circumscribed about the square and circle *B* is circumscribed about the equilateral triangle. Compute the ratio of the area of circle *A* to the area of circle *B*.

PART II ***FALL 2010*** ***CONTEST 4*** ***TIME: 10 MINUTES***

F10B21 Let $i = \sqrt{-1}$. Define a sequence where $a_1 = 2$ and $a_n = \begin{cases} a_{n-1} + i, & \text{if } n \text{ is even} \\ i \cdot a_{n-1}, & \text{if } n \text{ is odd} \end{cases}$.

Compute the value of the next term (after a_1) which is a real number.

F10B22 Let *ABCDEFGH* be a regular octagon. If the area of the octagon is 8, compute the area of rectangle *ABEF*.

PART III ***FALL 2010*** ***CONTEST 4*** ***TIME: 10 MINUTES***

F10B23 If Mary drives to work at an average speed of 40 miles per hour, she will arrive late by 1 minute. If she starts at the same time and drives the same route to work averaging 45 mph, she will arrive 1 minute early. Compute the distance in miles that Mary drives to work.

F10B24 Compute the sum of all two-digit numbers which have the property that the number equals seven more than the sum of the squares of its digits.

ANSWERS TO CONTEST 4

F10B19. (1003, -1002, 2)

F10B20. $\frac{27}{32}$

F10B21. -3

F10B22. 4

F10B23. 12

F10B24. 104

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 4

Fall 2010 Solutions

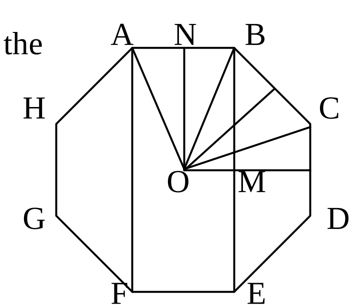
F10B19   **(1003, -1002, 2).** Since point (0, 2) is on the graph of the parabola, $c = 2$. Since (1, 3) is on the graph of the parabola 3 = *a* + *b* + 2 $\Rightarrow$ *a* + *b* = 1. Also, since (2, 2010) is on the graph of the parabola 2010 = 4*a* + 2*b* + 2 $\Rightarrow$ 2008 = 4*a* + 2*b* $\Rightarrow$ 2*a* + *b* = 1004. Subtracting *a* + *b* = 1 from 2*a* + *b* = 1004 yields *a* = 1003. Hence, *b* = –1002, and therefore, the ordered triple (*a*, *b*, *c*) is (1003, –1002, 2).

F10B20 $\mathbf{\frac{27}{32}}$. Since the square and the equilateral triangle have equal perimeters, for convenience let the side of the square = 3*x* and the side of the triangle = 4*x*. The radius of circle *A* is equal to half of the square's diagonal, or $\frac{3x\sqrt{2}}{2}$, and the radius of circle *B* is two-thirds of the triangle's altitude, or $\frac{2}{3}\left(2x\sqrt{3}\right) = \frac{4x\sqrt{3}}{3}$. Therefore, the ratio of the area of circle *A* to the area of circle *B* is

$$\left(\frac{\frac{3\sqrt{2}}{2}}{\frac{4\sqrt{3}}{3}}\right)^2 = \frac{\frac{18}{4}}{\frac{48}{9}} = \frac{9}{2}\cdot\frac{3}{16} = \frac{27}{32}.$$

F10B21   **-3**. If $a_1 = 2$, then $a_2 = 2 + i$, $a_3 = 2i - 1$, $a_4 = 3i - 1$, $a_5 = -3 - 1i$, and $a_6 = -3$. Hence, the answer is -3. (Challenge: what is the 2010th term of the sequence?)

F10B22.   **4**. Let the length of a side of the octagon be *x*. Diagonals $\overline{AF}$, $\overline{BE}$, $\overline{CH}$, and $\overline{DG}$ subdivide the octagon into four 45-45-90 triangles, 4 rectangles and 1 square. The legs of the 45-45-90 triangle are $\frac{x}{\sqrt{2}}$, so that the area of each triangle is $\frac{1}{2}\cdot\frac{x}{\sqrt{2}}\cdot\frac{x}{\sqrt{2}} = \frac{x^2}{4}$. The sides of each rectangle are *x* and $\frac{x}{\sqrt{2}}$, so that the area of each rectangle is $\frac{x^2}{\sqrt{2}}$. Next, the sides of the square are *x*, making the area of the square x^2. As a result, the area of the octagon can be expressed as $4\left(\frac{x^2}{4}\right) + 4\left(\frac{x^2}{\sqrt{2}}\right) + x^2 = 2x^2 + \frac{4x^2}{\sqrt{2}} = 2\left(x^2 + \frac{2x^2}{\sqrt{2}}\right)$. The area of rectangle *ABEF* can be expressed as $2\left(\frac{x^2}{\sqrt{2}}\right) + x^2 = \frac{2x^2}{\sqrt{2}} + x^2$. Finally, observe that the area of the rectangle is exactly half the area of the octagon, and therefore the area of *ABEF* is 4. You could also skip some of the area calculations by noticing that the total area of the four triangles equals the area of the square, and since *ABEF* contains the square and two of the smaller rectangles, its area must be exactly half that of the octagon.



Alternate solution: The octagon can be subdivided into 16 triangles by connecting the center to the eight vertices and drawing a perpendicular from the center to each side. Since the 16 triangles are all congruent, each one must have an area of ½. If we call the center *O*, the midpoint of $\overline{BE}$ is *M*, and the midpoint of $\overline{AB}$ is *N*, then we can show that $\Delta OBM \cong \Delta BON$. Since ΔBON is one of our original 16 triangles, ΔOBM must have area ½ as well. By repeating this procedure in all four quadrants of the octagon, or simply by noticing that ΔBON and ΔOBM

together form one quarter of rectangle *ABEF*, we can divide *ABEF* into exactly eight of these triangles, showing that its total area must be 4.

F10B23 **12**. It takes Mary 2 minutes longer to get to work traveling at 40 mph than it does traveling 45 mph. If we let D = the distance Mary travels to work, then using the idea that $\frac{\text{Distance}}{\text{Rate}} = \text{Time}$, we have $\frac{D}{40} = \frac{D}{45} + \frac{2}{60}$. Solving this equation for D yields $\frac{D}{40} = \frac{2D+3}{90}$ $\Rightarrow 90D = 80D + 120 \Rightarrow D = 12$.

F10B24 **104**. Let the two-digit number be $10A + B$. The property tells us that $10A + B = A^2 + B^2 + 7$. Rearranging this equation gives $A^2 - 10A + (B^2 - B + 7) = 0$, and by the quadratic formula, $A = \frac{10 \pm \sqrt{100 - 4(B^2 - B + 7)}}{2} = 5 \pm \sqrt{25 - (B^2 - B + 7)} = 5 \pm \sqrt{18 - B^2 + B}$. We need $\sqrt{18 - B^2 + B}$ to be an integer, and we know that B must be an integer from 0 to 9 inclusive. The only value of B which produces an integer is $B = 2$. So, with $B = 2$, we have $A = 5 \pm 4$ = 1, 9. Therefore, the solutions are 12 and 92, and their sum is 104.

Alternately, we can rewrite the equation $10A + B = A^2 + B^2 + 7$ as $10A - A^2 - 7 = B^2 - B$. For $A = 1, \ldots 9$, the values of the left side are 2, 9, 14, 17, 18, 17, 14, 9 and 2 in that order. For $B = 0, 1, \ldots 9$, the values of the right side are 0, 0, 2, 6, 12, 20, 30, 42, 56, and 72 in that order. The only value that both sides have in common is 2, when $B = 2$ and $A = 1$ or 9.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 5

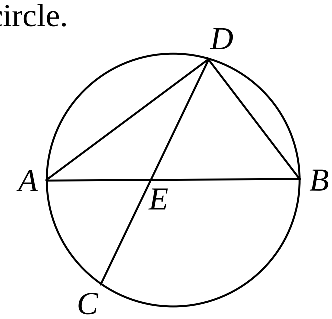
PART I | ***FALL 2010*** | ***CONTEST 5*** | ***TIME: 10 MINUTES***

F10B25   Compute $(\log_{125}16)(\log_4 27)(\log_3 625)$.

F10B26   Consider the linear function $f(x) = ax + b$. Compute all ordered pairs (a, b) such that for all real numbers x, $f(f(x)) + f(x) = 6x + 60$.

PART II | ***FALL 2010*** | ***CONTEST 5*** | ***TIME: 10 MINUTES***

F10B27   In the diagram below, $\overline{AD} \perp \overline{BD}$, $CD = 18$, $AE = 5$, and $CE = 8$. Compute the circumference of the circle.



F10B28   Define r and s to be the two roots of the equation $x^2 - 5x + 9 = 0$, and let the equation $ax^2 - bx + c = 0$ have roots $r + \frac{1}{s}$ and $s + \frac{1}{r}$. If a, b and c are relatively prime whole numbers, compute the ordered triple (a, b, c).

PART III | ***FALL 2010*** | ***CONTEST 5*** | ***TIME: 10 MINUTES***

F10B29   Compute the number of non-congruent isosceles triangles with a perimeter of 111 centimeters and all sides of integer length.

F10B30   Compute the number of three-digit numbers with three distinct digits in increasing or decreasing order. Only consider numbers whose leading digit is not zero.

ANSWERS TO CONTEST 5

F10B25.	8
F10B26.	(-3,-60) and (2,15)
F10B27.	21π
F10B28.	(9, 50, 100)
F10B29.	28
F10B30.	204

New York City Interscholastic Mathematics League

Senior B Division Contest Number 5

Fall 2010 Solutions

F10B25 **8**. Using the change of base formula and the power rule for logarithms, we can simplify the expression as follows: $\log_{125}16 \quad \log_4 27 \quad \log_3 625 = \frac{2\log 4}{3\log 5}\cdot\frac{3\log 3}{\log 4}\cdot\frac{4\log 5}{\log 3} = \frac{24}{3} = 8$.

F10B26 **(-3, -60) and (2, 15)**. We can substitute the value of $f(x)$ in terms of x into the equation to get $a(ax + b) + b + ax + b = 6x + 60$. Therefore, $(a^2 + a)x + ab + 2b = 6x + 60$, so $a^2 + a = 6 \Rightarrow a^2 + a - 6 = 0$ and $ab + 2b = 60 \Rightarrow b = \frac{60}{a+2}$. Solving the first of these equations gives us $a^2 + a - 6 = 0 \Rightarrow (a + 3)(a - 2) = 0 \Rightarrow a = -3, 2$. If $a = -3$, then $b = \frac{60}{-3+2} = -60$, and if $a = 2$, then $b = \frac{60}{2+2} = 15$. Hence, the ordered pairs are (–3, –60) and (2, 15).

F10B27 **21π**. Since $\overline{AD} \perp \overline{BD}$, $\overline{AB}$ is a diameter of the circle. By the intersecting chords theorem, $AE \cdot EB = CE \cdot ED$ and using substitution we have $5 \cdot EB = 8 \cdot 10 \Rightarrow EB = 16$. Since $AE = 5$ and $EB = 16$, $AB = 21$. Therefore the circumference of the circle is $AB\pi = 21\pi$.

F10B28 **(9, 50, 100)**. From $x^2 - 5x + 9 = 0$, we know that $r + s = 5$ and $rs = 9$. For the new equation, the sum of the roots is $\left(r + \frac{1}{s}\right) + \left(s + \frac{1}{r}\right) = (r + s) + \frac{r+s}{rs} = 5 + \frac{5}{9} = \frac{50}{9}$. Also, for the new equation the product of the roots is $\left(r + \frac{1}{s}\right)\left(s + \frac{1}{r}\right) = rs + \frac{1}{rs} + 2 = 9 + \frac{1}{9} + 2 = \frac{100}{9}$. As a result the new equation can be written as $9x^2 - 50x + 100 = 0$ and the ordered triple is (9, 50, 100).

F10B29 **28**. Since the perimeter is odd, the length of the base must be odd, so let the length of the base be $2x + 1$. The sum of the lengths of the legs is $111 - (2x + 1) = 110 - 2x$. By the triangle inequality theorem, $110 - 2x > 2x + 1 \Rightarrow 4x < 109 \Rightarrow x < 27\frac{1}{4}$. Therefore, x can have any value from 0 to 27, resulting in 28 possibilities. Alternately, if the legs have length A and the third side has length B, then $2A + B = 111$, and by the triangle inequality we have $2A > B$. Therefore $2A$ must be an even number in the range from 56 to 110, and A must be an integer in the range from 28 to 55, giving us 55 – (28 – 1) = 28 possibilities.

F10B30 **204**. For every 3 distinct digits selected from the set {1, 2, …, 9} there is exactly one way to arrange them into a number with increasing digits, and every number with increasing digits corresponds to one of these selections. Similarly, the numbers with decreasing digits correspond to the subsets with 3 elements of the set of all 10 digits. Hence, the answer is ${}_9C_3 + {}_{10}C_3 = \frac{9!}{6!\cdot 3!} + \frac{10!}{7!\cdot 3!} = \frac{9\cdot 8\cdot 7}{6} + \frac{10\cdot 9\cdot 8}{6} = 84 + 120 = 204$. Alternately, for every 3 distinct digits selected from the set {1, 2, …, 9} there is exactly one way to arrange them in increasing order and one way to arrange them in decreasing order, and for every set of three digits consisting of two digits from the set {1, 2, …, 9}and a zero, there is exactly one way to put them in decreasing order (we can't put them in increasing order because the zero would lead). Therefore, we can find the answer by calculating ${}_9C_3 \cdot 2 + {}_9C_2 =$

$$\frac{9!}{6! \cdot 3!} \cdot 2 + \frac{9!}{7! \cdot 2!} = \frac{9 \cdot 8 \cdot 7 \cdot 2}{6} + \frac{9 \cdot 8}{2} = 168 + 36 = 204.$$