



New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION
PART I: 10 minutes

CONTEST NUMBER ONE
NYCIML Contest One

FALL 2006
Fall 2006

F06B01 In $\triangle ABC$, $AC = 7$ and $BC = 9$. If h_A is the height drawn from vertex A and h_B is the height drawn from vertex B , compute $\frac{h_A}{h_B}$.

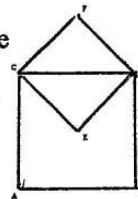
F06B02 If $k = \frac{(2004)^4 + 4(2004)^3(2) + 6(2004)^2(2)^2 + 4(2004)(2)^3 + (2)^4}{(2008)^3 - 3(2008)^2(2) + 3(2008)(2)^2 - (2)^3}$
Compute k .

PART II: 10 minutes

NYCIML Contest One

Fall 2006

F06B03 Square $ABDC$ has a side length of 2. $\overline{CD}$ is the diagonal of square $CEDF$ as shown, compute the length of $\overline{AF}$.



F06B04 If $x \cdot x^{\frac{1}{x}} = x^{2x}$ has three real roots, compute x .

PART III: 10 minutes

NYCIML Contest One

Fall 2006

F06B05 If the lines graphed by the equations $3x - 5y + 4 = 0$ and $2x + ay - 11 = 0$ meet at right angles, compute a .

F06B06 In Pascaland, there are only \$5 and \$11 bills. Compute the largest sum of money that cannot be made using these bills. (Assume that there exists an endless supply of each type of bill.)

ANSWERS

1. $\frac{7}{9}$
2. 2006
3. $\sqrt{10}$
4. $\pm 1, -\frac{1}{2}$
5. $\frac{6}{5}$
6. 39 (or \$39)



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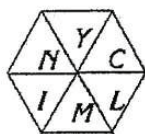
SENIOR B DIVISION

CONTEST NUMBER ONE

FALL 2006

Solutions

- For any triangle, $Area = \frac{1}{2}(\text{base})(\text{height})$. As such, any triangle has three base-height pairs. So, $A_{\triangle ABC} = \frac{1}{2}(AC)(h_B) = \frac{1}{2}(BC)(h_A)$. Thus, $\frac{(h_A)}{(h_B)} = \frac{(AC)}{(BC)} = \frac{7}{9}$.
- $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ and $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.
Thus, $k = \frac{(2004+2)^4}{(2008-2)^3} = \frac{(2006)^4}{(2006)^3} = 2006$.
- Construct midpoint G to $\overline{AB}$ and draw segments $\overline{GF}$ and $\overline{AF}$. $EF = 2$ and $GE = \frac{1}{2}(EF) = 1$. Thus, $GF = 3$ and $AG = 1$. By the Pythagorean Theorem,
 $AF = \sqrt{(AG)^2 + (GF)^2} = \sqrt{10}$.
- We can simplify the exponent on the left hand side as follows:
 $x \cdot x^{\frac{1}{x}} = x^1 \cdot x^{\frac{1}{x}} = x^{1+\frac{1}{x}} = x^{\frac{x+1}{x}}$. Thus, by equating the exponents, we get
 $\frac{x+1}{x} = 2x \rightarrow x+1 = 2x^2$. Solving this quadratic yields $x = -\frac{1}{2}$ and $x = 1$. The question said that there are 3 real roots, so a careful look at the problem leads us to see that $x = -1$ also works. Thus the roots are $\pm 1, -\frac{1}{2}$.
- If the two lines meet at right angles, their slopes are negative reciprocals. Putting both lines into slope-intercept form yields $y = \frac{3}{5}x + \frac{4}{5}$ and $y = -\frac{2}{a}x + \frac{11}{a}$.
So, $\frac{3}{5} = \frac{a}{2} \rightarrow a = \frac{6}{5}$.
- Let x = number of \$5 bills, and y = number of \$11 bills. One way to proceed is to realize that there are two ways to make \$55-- (11,0) and (0,5). Moreover, we can "manufacture" \$1 by subtracting two \$5's and adding one \$11. Thus, \$56, \$57, \$58, and \$59 are (9,1), (7,2), (5,3), and (3,4) respectively. We can now make \$60 out of \$5's entirely and then proceed to all higher multiples of 5 in a similar manner. In fact, the same process can begin at \$45 with (9,0) and at \$40 with (8,0). A problem happens at \$35 with (7,0). There are not enough \$5's to exchange to create \$39 which would have the solution (-1,4). Thus, \$39 is the answer.



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SENIOR B DIVISION
PART I: 10 minutes

CONTEST NUMBER TWO
NYCIML Contest Two

FALL 2006
Fall 2006

F06B07 If $\log(8) + \log(\sqrt[1]{8}) = x$, compute x .

F06B08 If $x + y + z = 7$ and $x^2 + y^2 + z^2 = 10$, compute $xy + xz + yz$.

PART II: 10 minutes

NYCIML Contest Two

Fall 2006

F06B09 A circle is inscribed in a regular hexagon. The perimeter of the hexagon is 36, compute the area of the circle.

F06B10 If $2006jc$ is a 6-digit base 10 number (the digits are 2, 0, 0, 6, j , c) that is divisible by 72, compute (j, c) .

PART III: 10 minutes

NYCIML Contest Two

Fall 2006

F06B11 $\frac{a}{b}$ is in lowest terms. If the denominator of the fraction is added to both the numerator and the denominator, the fraction is doubled. Compute $\frac{a}{b}$.

F06B12 If all the 6-digit numbers formed by using the digits 1, 2, 3, 4, 5, and 6, without repetition, are listed from least to greatest, compute the 500th number.

ANSWERS

- 7. -1
- 8. $\frac{39}{2}$
- 9. 27π
- 10. (6, 4)
- 11. $\frac{1}{3}$
- 12. 516243



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CONTEST NUMBER TWO

FALL 2006

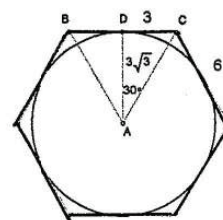
Solutions

$$7. \quad \log(8) \div \log\left(\frac{1}{8}\right) = \frac{\log(8)}{-1 \cdot \log(8)} = -1.$$

$$8. \quad (x+y+z)^2 = 49 = x^2 + y^2 + z^2 + 2(xy+xz+yz). \text{ Thus,}$$

$$49 = 10 + 2(xy+xz+yz). \text{ This yields } xy+xz+yz = \frac{39}{2}.$$

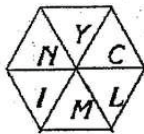
9. Since the perimeter of the hexagon is 36, each side is 6. $\overline{AD}$ is an angle bisector, a median and an altitude to $\overline{BC}$. Thus $CD = 3$ and triangle ABC is a 30, 60, 90 triangle. $AD = 3\sqrt{3}$. The area of the circle is $\pi \cdot (3\sqrt{3})^2 = 27\pi$.



10. If a number is divisible by 72, then it must satisfy divisibility by 8 and 9. For divisibility by 8, the last three digits of the number must form a 3-digit number that is a multiple of 8. For divisibility by 9, the sum of the digits of the number must be a multiple of 9 therefore $8+j+c=9$ or $18 \rightarrow j+c=1$ or 10 . To test to see if $6jc$ is a multiple of 8, the only numbers we must try to divide by 8 are 682, 664, 646, 628, 610. Only 664 satisfies divisibility by 8 and the answer is 200664.

$$11. \quad \frac{a+b}{2b} = \frac{2a}{b} \rightarrow \frac{a+b}{2a} = \frac{2b}{b} \rightarrow \frac{a+b}{2a} = 2 \rightarrow a+b = 4a \rightarrow b = 3a \rightarrow \frac{a}{b} = \frac{1}{3}.$$

12. There are a total of $6! = 720$ permutations of these digits. Of these, $5! = 120$ begin with "1", 120 begin with "2", and so forth. Thus, the last number that begins with "4" is the 480th number. $4! = 24$ numbers begin with "51", so the 504th number is 516432. There are $3! = 6$ numbers that begin with "512", 6 numbers that begin with "513", and 6 numbers that begin with "514". Thus the 498th number is 514632. 499th is 516234. 500th is 516243.



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PART I: 10 minutes

CONTEST NUMBER THREE
NYCIML Contest Three

FALL 2006
Fall 2006

F06B13 Compute all real x : $x + \sqrt{x-5} = 7$.

F06B14 Compute the unit's digit of: $1^5 + 2^5 + 3^5 + 4^5 + 5^5 + 6^5 + \dots + 2006^5$.

PART II: 10 minutes

NYCIML Contest Three

Fall 2006

F06B15 In $\triangle ABC$, $\sin(A) = \frac{3}{5}$ and $\sin(B) = \frac{4}{5}$. Compute the value of $\sin(C)$.

F06B16 Larry and Gill are going to play a game. They put cards with the letters C, O, R, N, E, L , and L in a hat (each of the seven cards has one of the letters and there are two cards with an L .) They will alternate picking cards, one at a time, without replacement, until someone wins by picking an L . (They do not see the card they pick until they pick it.) If Larry picks first, compute the probability that Larry will win.

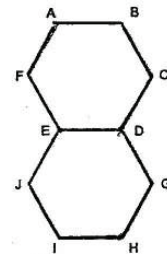
PART III: 10 minutes

NYCIML Contest Three

Fall 2006

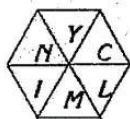
F06B17 If $3x^2 - 3x + h$ is divisible by $x - 4$, compute h .

F06B18 $ABCDEF$ and $EDGHIJ$ are regular hexagons as shown. If $AB = 2$, compute AH .



ANSWERS

13. **6**
14. **1**
15. **1**
16. **$\frac{4}{7}$**
17. **-36**
18. **$2\sqrt{13}$**



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CONTEST NUMBER THREE

FALL 2006

Solutions

13. Rearranging the equation we get $\sqrt{x-5} = 7-x$. Squaring both sides gives $x^2 - 15x + 54 = 0$. This yields two solutions, $x = 6$ and $x = 9$. Only $x = 6$ satisfies the original equation.

14. It can be shown that the unit's digit of x^5 , where x is a base-10 number, is the same as the unit's digit of x . Thus, $1^5 + 2^5 + 3^5 + 4^5 + 5^5 + 6^5 + \dots + 2006^5$ has the same unit's digit as $1 + 2 + 3 + 4 + \dots + 9 + 0 + 1 + 2 + \dots + 5 + 6$. The unit's digit of $1 + 2 + 3 + 4 + \dots + 9 + 0$ is 5. Thus, the unit's digits of the sum every 20 consecutive integers is 0. As a result, the sum of the first 2000 numbers has a unit's digit of 0. Therefore, the unit's digit of the sum in this problem is just the unit's digit of $1 + 2 + 3 + 4 + 5 + 6$, which is 1.

15. In a 3-4-5 right triangle, the value of the sine of the acute angles are $3/5$ and $4/5$. Thus, $\angle A$ and $\angle B$ are complementary. $\angle C$ is a right angle, and $\sin(C) = 1$.

16. Larry can win on the 1st pick, the 3rd pick, or the 5th pick. Larry would never have the opportunity to have a 7th pick because if Gill had an opportunity to have a turn on the 6th pick, he would be guaranteed to pick an "L" because the two remaining letters would both be "L". Thus, $P(\text{Larry wins}) =$

$$\left(\frac{2}{7}\right) + \left(\frac{5}{7}\right)\left(\frac{4}{6}\right)\left(\frac{2}{5}\right) + \left(\frac{5}{7}\right)\left(\frac{4}{6}\right)\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{2}{3}\right) = \frac{12}{21} = \frac{4}{7}.$$

17. One way to attack this problem is to use the Remainder Theorem. Thus, if $f(x) = 3x^2 - 3x + h$ is divisible by $x - 4$, then $f(4) = 3(4)^2 - 3(4) + h = 0$. Solving for h yields $h = -36$.

18. After we compute AI , we can use the Pythagorean Theorem in $\triangle AIH$ to find AH . Since $AI = 2(AE)$, we need to find AE . To accomplish this, we can use the law of cosines and the fact that $m\angle AFE = 120^\circ$.

$$(AF)^2 + (FE)^2 - 2(AF)(FE)\cos(120^\circ) = (AE)^2$$

Substituting for AF and FE yields $(2)^2 + (2)^2 - 2(2)(2)\left(-\frac{1}{2}\right) = (AE)^2$. Therefore,

$AE = 2\sqrt{3}$ and $AI = 4\sqrt{3}$. We now use $(AI)^2 + (IH)^2 = (AH)^2$. Thus,

$$AH = \sqrt{(4\sqrt{3})^2 + (2)^2} = \sqrt{52} = 2\sqrt{13}.$$



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SENIOR B DIVISION
PART I: 10 minutes

CONTEST NUMBER FOUR
NYCIML Contest Four

FALL 2006
Fall 2006

- F06B19** If $i = \sqrt{-1}$, and the value of $(1-i)^{10} = a + bi$ where a and b are integers, compute (a, b) .
- F06B20** The average (arithmetic mean) of a set of h numbers is m . The average of a different set of m numbers is h . If the average of all the numbers is p , express $\frac{1}{h} + \frac{1}{m}$ in terms of p , in simplest form.

PART II: 10 minutes

NYCIML Contest Four

Fall 2006

- F06B21** Compute the number of integers from 1 to 2006 inclusive that are not divisible by 3 or 7.
- F06B22** In $\triangle ABC$, point D lies on $\overline{AC}$ such that $AD < DC$. If $AB = 20$, $AC = 30$, $BD = 13$, and the area of $\triangle ABC = 180$, compute the area of $\triangle BDC$.

PART III: 10 minutes

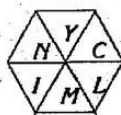
NYCIML Contest Four

Fall 2006

- F06B23** Siggy tosses 2 fair coins. Siggy shows you one of the coins, and it is "heads". Compute the probability that the other coin is "tails"?
- F06B24** Compute all ordered triples (x, y, z) of integers:
- $$\begin{aligned}x + yz &= 6 \\y + xz &= 6 \\z + xy &= 6\end{aligned}$$

ANSWERS

19. $(0, -32)$
20. $\frac{2}{p}$
21. 1147
22. 114
23. $\frac{2}{3}$
24. $(1, 1, 5), (1, 5, 1), (5, 1, 1), (2, 2, 2),$ and $(-3, -3, -3)$



Solutions

19. $(1-i)^2 = 1-2i+i^2 = -2i$. Thus, $(1-i)^{10} = (-2i)^5 = -32i^5 = -32i$. Therefore, $a = 0$, and $b = -32$ and the answer is $(0, -32)$.

20. The sum of each set of numbers is hm ; therefore, the sum of all $h+m$ numbers is $2hm$. So, $p = \frac{2hm}{h+m}$. We can now obtain $\frac{1}{p} = \frac{h+m}{2hm} = \frac{1}{2}\left(\frac{1}{m} + \frac{1}{h}\right)$. Thus, $\frac{1}{h} + \frac{1}{m} = \frac{2}{p}$.

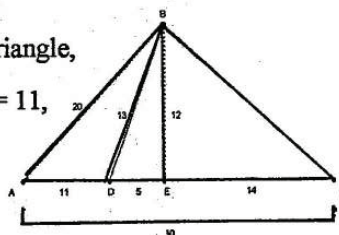
21. There are $\left\lfloor \frac{2006}{3} \right\rfloor = 668$ multiples of 3 less than or equal to 2006, and

$\left\lfloor \frac{2006}{7} \right\rfloor = 286$ multiples of 7. However, there are $\left\lfloor \frac{2006}{21} \right\rfloor = 95$ multiples of 21 that have been counted in both lists. Therefore, there are $668 + 286 - 95 = 859$ numbers that have 3 or 7 as a factor, and as a result, there are $2006 - 859 = 1147$ numbers that have neither 3 nor 7 as a factor. ($\lfloor x \rfloor$ represents the greatest integer less than or equal to x .)

22. Draw altitude $\overline{BE}$. From the area of $\triangle ABC$, we know that

$180 = \frac{1}{2}(AC)(BE) = 15(BE)$. Thus, $BE = 12$. $\triangle BDE$ is a right triangle, so $DE = 5$. $\triangle BAE$ is a right triangle, so $AE = 16$. Therefore, $AD = 11$, and $DC = 19$. Thus, the area of $\triangle BDC = \frac{1}{2}(19)(12) = 114$.

Note, that if point D were between points E and C , then AD would equal 21 and DC would equal 9, thereby violating the constraint requiring $AD < DC$.



23. The possible outcomes are HH , HT , TH , and TT . We know that one of the two coins is H , so the outcome TT is impossible. Of the three outcomes remaining, two have "tails". Thus, the probability is $\frac{2}{3}$.

24. If we subtract the first two equations, we obtain $x - y + yz - xz = (x-y)(1-z) = 0$.

Thus, either $x = y$ or $z = 1$. If $x = y$, then: $x + xz = 6$
 $z + x^2 = 6$. Substituting for z yields

$x + x(6 - x^2) = 6$ or $x^3 - 7x + 6 = 0$. We know x must be an integer, so we can use the rational roots theorem to test the eight possibilities, namely $\pm 1, \pm 2, \pm 3, \pm 6$. Of these, $x = 1, 2$ and -3 work. This leads to solutions $(1, 1, 5)$, $(2, 2, 2)$ and $(-3, -3, -3)$. Noting that the original system has symmetry, we can conclude that the remaining two solutions are $(1, 5, 1)$ and $(5, 1, 1)$. We could also get these remaining two solutions by using $z = 1$ and solving for x and y . The required triples are: $(1, 1, 5)$, $(1, 5, 1)$, $(5, 1, 1)$, $(2, 2, 2)$, and $(-3, -3, -3)$.



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SENIOR B DIVISION
PART I: 10 minutes

CONTEST NUMBER FIVE
NYCIML Contest Five

FALL 2006
Fall 2006

- F06B25** Jimmy has a 10 ounce drink that is 60% coffee. Jimmy adds milk until his drink is 40% coffee. Compute the amount of milk, in ounces, that he added.
- F06B26** If $h = 11!$, compute the number of positive integral divisors of h .
-

PART II: 10 minutes

NYCIML Contest Five

Fall 2006

- F06B27** Helen rolls one fair six-sided die. Jim rolls two fair six-sided dice. Compute the probability that the sum of the numbers showing on Jim's dice equals the number showing on Helen's die.
- F06B28** If $x^3 + 2x^2 + kx + 3 = 0$ has at least one rational root, compute all possible values of k .
-

PART III: 10 minutes

NYCIML Contest Five

Fall 2006

- F06B29** If $4 - \frac{3}{2 + \frac{x}{1-x}} = \frac{a-x}{b-x}$, $x \neq 1, 2$, a and b are integers, compute (a, b) .
- F06B30** In rectangle $ABCD$, point E is the intersection of the two diagonals, point G lies on $\overline{BC}$, and point F is the intersection of $\overline{AC}$ and $\overline{DG}$. If $AB = 40$, $BC = 30$, and $BG = 10$, compute EF .
-

ANSWERS

25. 5
26. 540
27. $\frac{5}{72}$
28. -16, -6, -2 and 4
29. (5, 2)
30. 5



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CONTEST NUMBER FIVE

FALL 2006

Solutions

25. Currently, Jimmy has 6 oz of coffee. If x is the amount of milk added, then:

$$\frac{6}{10+x} = \frac{2}{5} \rightarrow 30 = 20 + 2x \rightarrow x = 5.$$

26. To determine the number of factors, we need the prime factorization of $11!$. This yields $2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1 \cdot 11^1$. Thus, there are $(8+1)(4+1)(2+1)(1+1)(1+1) = 540$.

27. Helen can have outcomes 1, 2, 3, 4, 5 and 6. Jim can have outcomes 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. Thus, the only overlaps are 2, 3, 4, 5 and 6. The probability of a match is therefore $P(\text{both 2}) + P(\text{both 3}) + P(\text{both 4}) + P(\text{both 5}) + P(\text{both 6}) =$

$$\left(\frac{1}{6}\right)\left(\frac{1}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{2}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{3}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{4}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{36}\right) = \frac{15}{216} = \frac{5}{72}.$$

28. If the polynomial has a rational root, and k is an integer, then the rational roots theorem states that the only possible rational roots are ± 1 and ± 3 . Substituting these values for x , the possible values of k are **-6, 4, -16 and -2**.

29. $4 - \frac{3}{2 + \frac{x}{1-x}} = 4 - \frac{3-3x}{2-2x+x} = 4 - \frac{3-3x}{2-x} = \frac{8-4x}{2-x} - \frac{3-3x}{2-x} = \frac{5-x}{2-x} = \frac{a-x}{b-x}$. Thus

$$(a, b) = (5, 2).$$

30. $\triangle CGF$ is similar to $\triangle ADF$. Therefore, $\frac{GC}{AD} = \frac{CF}{AF}$. $AC = 50$ so $AE = EC = 25$.

Let $EF = x$, then the proportion becomes $\frac{20}{30} = \frac{25-x}{25+x} \rightarrow 50 + 2x = 75 - 3x \rightarrow x = 5$.

