NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 1

PART I *SPRING 2006* *CONTEST 1* *TIME: 10 MINUTES*

S06A1 The price of Endrun's stock increased by 20% at the end of the first year, then by 25% of the new price at the end of the second year and then by 30% of the second year's price at the end of the third year. In the fourth year the stock lost 80% of its third year price. If k represents the ratio of the price of the stock at the end of the fourth year to the original price of the stock, compute k.

S06A2 A rectangular sheet of paper measures 9 inches by 12 inches. The paper is folded once such that the opposite vertices share a common point. Compute the length of the fold in the paper.

PART II *SPRING 2006* *CONTEST 1* *TIME: 10 MINUTES*

S06A3 Compute the sum of the roots of the equation: $||x-3|-5|=3$.

S06A4 If $a \neq b$ and a, b, x, y are positive integers, compute, in terms of a and b, the number of ordered pairs (x, y) that satisfy: $x^2+y^2+2xy-(a+b)x-(a+b)y+ab=0$.

PART III *SPRING 2006* *CONTEST 1* *TIME: 10 MINUTES*

S06A5 A lottery consists of picking five numbers out of 54 distinct numbers. Five "winning" numbers are chosen randomly. First prize is won by getting all five winning numbers. Second prize is won by getting exactly four out of five. Compute the ratio of the probability of winning a second prize to the probability of winning a first prize.

S06A6 $P(x)$ is a polynomial with integral coefficients. When $P(x)$ is divided by $x-3$, the remainder is 25. When $P(x)$ is divided by $x+3$, the remainder is 7. Compute the remainder when $P(x)$ is divided by x^2-9.

ANSWERS:

S06A1	$\frac{39}{100}$ **or .39**
S06A2	**11.25 or** $\frac{45}{4}$
S06A3	**12**
S06A4	$a+b-2$
S06A5	**245**
S06A6	**3x + 16**

New York City Interscholastic Mathematics League

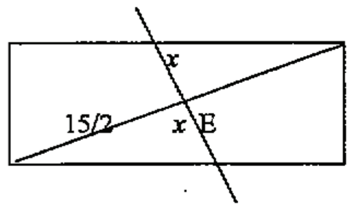
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 1

SPRING 2006 Solutions

S06A1 Let the original price =\$100. The current price = \$100(100+20)%(100+25)%(100+30)%(100-80)% = 100(1.2)(1.25)(1.3)(.2) = \$39
\$39/\$100=39/100 or .39

S06A2



The diagonal= $\sqrt{9^2+12^2}=15$. E is the point of intersection between the diagonal and the fold. The fold is the perpendicular bisector of the diagonal. Let the fold =2x. $\dfrac{x}{15/2}=\dfrac{9}{12}\therefore x=\dfrac{45}{8}in$.

$2x=\dfrac{\mathbf{45}}{\mathbf{4}}=11.25$ inches

S06A3 The equation simplifies to $|x-3|-5=3$ or $-|x-3|+5=3$.
If $|x-3|=8$ then $x=11,-5$. If $|x-3|=2$ then $x=1,5$. The sum of the roots is **12**.

S06A4 $x^2+y^2+2xy-(a+b)x-(a+b)y+ab=0$ simplifies to $(x+y)^2-(a+b)(x+y)+ab=0$
We then factor $(x+y-a)(x+y-b)=0$. $x+y-a=0$ or $x+y-b=0$ so $x+y=a$ or b
There are a -1 solutions if $x+y=a$ and $b-1$ solutions if $x+y=b$ $\therefore \mathbf{a+b-2}$ solutions.

S06A5 $P(winning\ 2nd)=\dfrac{\binom{5}{4}\binom{49}{1}}{\binom{54}{5}}$ $P(winning\ 1st)=\dfrac{\binom{5}{5}}{\binom{54}{5}}\therefore \dfrac{P(2nd)}{P(1st)}=\binom{5}{4}\binom{49}{1}=\mathbf{245}$

S06A6 $P(x)=(x-3)\cdot s(x)+25=(x+3)\cdot t(x)+7=(x^2-9)\cdot q(x)+r(x)$
$P(3)=25=r(3)$ and $P(-3)=7=r(-3)$. Because we are dividing by a quadratic, the remainder $r(x)=mx+b$. $m=\dfrac{\Delta y}{\Delta x}=\dfrac{18}{6}=3$. $r(3)=25=3(3)+b\therefore b=16$
$r(x)=\mathbf{3x+16}$

New York City Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior A Division CONTEST NUMBER 2

PART I *SPRING 2006* *CONTEST 2* *TIME: 10 MINUTES*

S06A7 Compute the smallest value of n for which a regular polygon of n sides has at least 2006 diagonals.

S06A8 If $n = 30^4$, then the sum of the positive integral divisors of n may be expressed as a^3bc, where a, b and c are prime. Compute $a + b + c$.

PART II *SPRING 2006* *CONTEST 2* *TIME: 10 MINUTES*

S06A9 A square, with side of length s, is inscribed in an equilateral triangle, with side of length t, such that two vertices of the square are on one side of the triangle. The other vertices are on the remaining two sides. The ratio $s : t$ may be written as $1 : a + b\sqrt{3}$. Compute $a + b$.

S06A10 If the roots of $x^3 - ax^2 - bx - c = 0$ are a, b, and c, compute the ordered triple of numbers (a, b, c). (a, b, c, are all non zero)

PART III *SPRING 2006* *CONTEST 2* *TIME: 10 MINUTES*

S06A11 Let $f(x) = ax^7 + bx^5 + cx^3 + x + 4$ where a, b, and c are real numbers. If $f(11) = 17$, compute $f(-11)$.

S06A12 If $(x+y):(y+z):(x+z) = 1:2:4$ and $x+y+z = 35$, compute the value of x.

ANSWERS:

S06A7	**65**
S06A8	**113**
S06A9	$\frac{5}{3}$
S06A10	**(1, 1, -1)**
S06A11	**-9**
S06A12	**15**

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NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 2

SPRING 2006 Solutions

S06A7 The number of diagonals = the total number of ways to connect n points two at a time – the number of sides = ${}_nC_2 - n$. ${}_nC_2 - n \geq 2006$, $\frac{n(n-1)}{2} - n \geq 2006$

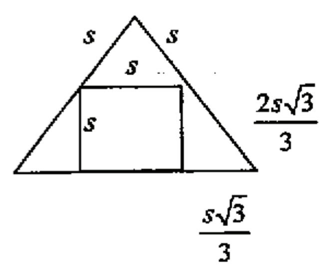
$n^2 - 3n \geq 4012$ and $n(n-3) \geq 4012 \therefore n = \mathbf{65}$

S06A8 $30^4 = (2\cdot 3\cdot 5)^4 = 2^4 3^4 5^4$

The sum of the divisors $= (1+2+2^2+2^3+2^4)(1+3+3^2+3^3+3^4)(1+5+5^2+5^3+5^4)$

$= (31)(121)(781) = 31(11)(11)(11)(71) = 11^3\cdot 31\cdot 71 \therefore a+b+c = 11+31+71 = \mathbf{113}$

S06A9



The top triangle is equilateral, $\therefore$ all the sides are s.
The bottom triangles are 30-60-90 triangles

Therefore the sides are in proportion $1:\sqrt{3}:2$

$s:t = 1:\left(1+\frac{2\sqrt{3}}{3}\right) \therefore a = 1, b = \frac{2}{3}.\ \ a+b = \frac{5}{3}$

S06A10 The product of the roots = $abc = c$, therefore $ab = 1$. The sum of the roots = $a + b + c = a$, hence: $b + c = 0$ or $b = -c$. The sum of the roots taken two at a time = $ab + bc + ac = -b$. $(1) + bc + (-1) = -b \ \therefore c = -1,\ b = 1$ and $a = 1$. The answer is **(1, 1, -1).**

S06A11 $f(11) = a(11)^7 + b(11)^5 + c(11)^3 + (11) + 4 = 17$

$f(-11) = a(-11)^7 + b(-11)^5 + c(-11)^3 + (-11) + 4 = -f(11) + 8 = \mathbf{-9}$

S06A12

$(x+y):(y+z) = 1:2$	$(x+y):(x+z) = 1:4$
$2x+2y = y+z$	$4x+4y = x+z$
$+x \quad = \quad +x$	$+y \quad = \quad +y$
$3x+2y = x+y+z = 35$	$4x+5y = x+y+z = 35$

Solving $3x+2y = 35$ and $4x+5y = 35$ simultaneously gives us $x = \mathbf{15}$.

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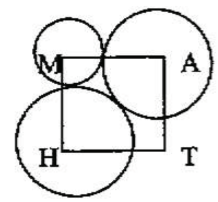
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 3

PART I ***SPRING 2006*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S06A13 Compute the sum of the distinct prime factors of: $3^{12}-1$.

S06A14 Each side of square MATH has a length of 2. M, A, and H are the centers of three mutually, externally tangent, circles. Compute the radius of the smallest of these circles.



PART II ***SPRING 2006*** ***CONTEST 3*** ***TIME: 10 MINUTES***

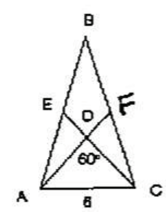
S06A15 If: $\frac{\sin 1°}{\cos 1°} \cdot \frac{\sin 2°}{\cos 2°} \cdot \frac{\sin 3°}{\cos 3°} \cdots \frac{\sin 88°}{\cos 88°} \cdot \frac{\sin 89°}{\cos 89°} = \tan k°$, compute k.

S06A16 $\sqrt[3]{x+2} - \sqrt[3]{x-2} = 1$, compute x^2.

PART III ***SPRING 2006*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S06A17 A motorboat has a speed of 30 mph when traveling in a river with no current. The boat travels up and down a river that has a current of 10 mph. Compute the ratio of the average speed for the round trip on the river with the current to the average speed for the same trip if there were no current.

S06A18 The medians to the legs of isosceles triangle ABC intersect to form a 60° angle as shown. If the base of the triangle is 6, compute the area of triangle ABC.



ANSWERS:		
	S06A13	**100**
	S06A14	$2-\sqrt{2}$
	S06A15	**45**
	S06A16	**5**
	S06A17	$\frac{8}{9}$
	S06A18	$27\sqrt{3}$

New York City Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

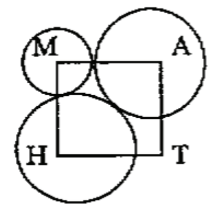
Senior A Division CONTEST NUMBER 3

SPRING 2006 Solutions

S06A13 $3^{12}-1=(3^6-1)(3^6+1)=730\cdot 728=(10\cdot 73)(4\cdot 182)=2\cdot 5\cdot 73\cdot 2\cdot 2\cdot 2\cdot 7\cdot 13$

The sum of the distinct prime factors of $3^{12}-1=2+5+7+13+73=\mathbf{100}$

S06A14 $AH=\ 2\sqrt{2}$ Due to the symmetry, the circles at A and H have the same radius $R=\frac{AH}{2}=\sqrt{2}$. The circle at M Has a radius $r=MA-R=\mathbf{2-\sqrt{2}}$



S06A15 $\sin\theta=\cos(90-\theta)\quad \therefore \frac{\sin 1°}{\cos 1°}\frac{\sin 2°}{\cos 2°}\frac{\sin 3°}{\cos 3°}\cdots\frac{\sin 88°}{\cos 88°}\frac{\sin 89°}{\cos 89°}=$

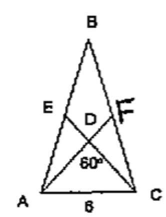
$\frac{\sin 1°}{\sin 89°}\frac{\sin 2°}{\sin 88°}\frac{\sin 3°}{\sin 87°}\cdots\frac{\sin 88°}{\sin 2°}\frac{\sin 89°}{\sin 1°}=1\therefore\ \tan k°=1\therefore k=\mathbf{45}$

S06A16 Let $a=\sqrt[3]{x+2}$ and $b=\sqrt[3]{x-2}$. $a-b=1$ and $(a-b)^3=1\rightarrow a^3-b^3-3ab(a-b)=1$. From this we get $4-3ab=1\rightarrow ab=1$. Now: $\sqrt[3]{x^2-4}=1\rightarrow x^2=\mathbf{5}$.

S06A17 speed · time=distance $\frac{\text{speed with current}}{\text{speed without current}}=\frac{\text{time without current}}{\text{time with current}}$

Without a loss of generality, we can pick any distance for the trip. If d = 120 miles, the time without the current is 240 / 30 = 8. With the current, the speed downstream is 40, taking 3 hours to travel 120 miles. With the current, the speed upstream is 20, taking 6 hours to travel 120 miles. $\therefore$ the answer is $\mathbf{\frac{8}{9}}$.

S06A18 Let the area of triangle $ADC=K$, then the area of triangle ACE is $\frac{3}{2}K$ because the medians of a triangle divide each other in a 2:1 ratio and therefore $CD:DE$ is 2:1.(The base EC is 3/2 times the base CD.) Area of triangle ABC is twice the area of triangle $ACE=3K$. (The base AB is twice the base AE.) Triangle ADC is equilateral with side 6 so $K=\frac{s^2\sqrt{3}}{4}=9\sqrt{3}$ and $3K=\mathbf{27\sqrt{3}}$.



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NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 4

PART I	*SPRING 2006*	*CONTEST 4*	*TIME: 10 MINUTES*

S06A19 Compute the sum of the following series:
$3+\frac{3}{4}\sqrt{3}+\frac{3}{4}+\frac{3}{16}\sqrt{3}+\frac{3}{16}+\cdots$

S06A20 If: $0 < A < 90^\circ$ and $2\cos A = \sqrt{3}\cos 23^\circ - \sin 23^\circ$, compute A.

PART II	*SPRING 2006*	*CONTEST 4*	*TIME: 10 MINUTES*

S06A21 A box contains marbles and each marble is numbered once. One marble has the number 1, two marbles have the number 2, and so on until the last 2006 marbles are numbered 2006. One marble is drawn randomly. The probability, in simplest form, that the marble has an odd number is $\frac{p}{q}$. Compute $p + q$.

S06A22 Let k be an odd number greater than 1. Compute the minimum number of k consecutive positive integers that have a sum of 2006.

PART III	*SPRING 2006*	*CONTEST 4*	*TIME: 10 MINUTES*

S06A23 If $\log(\sec x) - \log(\cos x) = 1$ and $0 < x < \frac{\pi}{2}$, compute $\sin x$.

S06A24 Rectangle $ABCD$ has a length of 8 inches and a width of 6 inches. Triangle BCD is reflected over diagonal $\overline{BD}$ such that C' is the image of C after the reflection. Compute AC'.

ANSWERS:

S06A19	$\mathbf{4+\sqrt{3}}$
S06A20	**53°**
S06A21	**3010**
S06A22	**17**
S06A23	$\mathbf{\frac{3\sqrt{10}}{10}}$
S06A24	$\frac{14}{5}$ **or 2.8 or** $2\frac{4}{5}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 4

SPRING 2006 Solutions

S06A19

$$S = 3 + \frac{3}{4}\sqrt{3} + \frac{3}{4} + \frac{3}{16}\sqrt{3} + \frac{3}{16} + \cdots$$

$$S + 3\sqrt{3} = 3\sqrt{3} + 3 + \frac{3}{4}\left(\sqrt{3}+1\right) + \frac{3}{16}\left(\sqrt{3}+1\right) + \cdots = \frac{\left(3+3\sqrt{3}\right)}{1-\frac{1}{4}} = 4\left(1+\sqrt{3}\right) = 4 + 4\sqrt{3}$$

$$S = \mathbf{4 + \sqrt{3}}$$

S06A20

$$2\cos A = \sqrt{3}\cos 23^\circ - \sin 23^\circ$$

$$\cos A = \frac{\sqrt{3}}{2}\cos 23^\circ - \frac{1}{2}\sin 23^\circ = \cos 30^\circ \cos 23^\circ - \sin 30^\circ \sin 23^\circ$$

$\cos(x+y) = \cos x \cos y - \sin x \sin y \therefore \cos A = \cos(30^\circ + 23^\circ)$ and $A = \mathbf{53^\circ}$

S06A21

$$P(odd\ \#) = \frac{\#\text{ of odds}}{\text{Total }\#\text{ of marbles}} = \frac{1+3+5+\cdots+2005}{1+2+3+\cdots+2006} = \frac{\frac{1003(2006)}{2}}{\frac{2006(2007)}{2}} = \frac{1003}{2007} \therefore p+q = \mathbf{3010}.$$

S06A22

Let the consecutive integers be $a, a+1, a+2, \ldots, a+(k-1)$ where a is the first term and k is the number of terms. The sum, $S = \frac{k}{2}(2a+k-1) = k\left(a + \frac{k-1}{2}\right) = 2006$. The odd factors of 2006 are 1, 17, 59, 1003 so the minimum $k = \mathbf{17}$ ($a = 110$).

S06A23

$$\log \sec x = \log \frac{1}{\cos x} = -\log \cos x, \qquad -\log \cos x - \log \cos x = 1 \therefore \log \cos x = \frac{-1}{2}$$

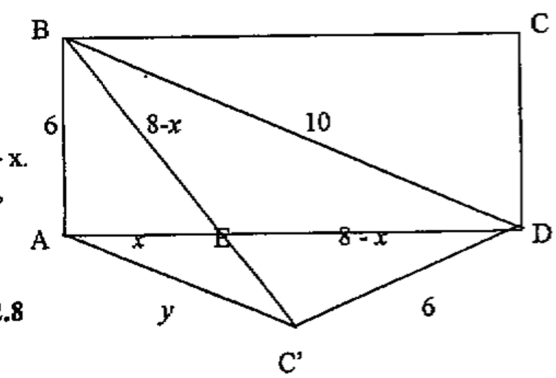
$$\cos x = \frac{1}{\sqrt{10}} \therefore \cos^2 x = \frac{1}{10} \text{ and } \sin^2 x = \frac{9}{10},\ \sin x = \sqrt{\frac{9}{10}} = \mathbf{\frac{3\sqrt{10}}{10}}$$

S06A24

When we reflect C over line BD, the shape $ABDC'$ is an isosceles trapezoid. If we call E the pt of intersection of lines AD and BC', then $AE = EC' = x$ and $ED = EB = 8 - x$. Solve for x, using right triangle ABE,

$$6^2 + x^2 = (8-x)^2 \therefore x = \frac{7}{4}$$

$$\frac{y}{10} = \frac{x}{8-x} \text{ or } \frac{y}{10} = \frac{7/4}{25/4} \therefore y = \frac{14}{5} = \mathbf{2.8}$$



New York City Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 5

PART I *SPRING 2006* *CONTEST 5* *TIME: 10 MINUTES*

S06A25 Compute the numerical coefficient of $ab^2c^3d^4$ when $(a+b+c+d)^{10}$ is expanded.

S06A26 Two swimmers, start at opposite ends of a 100-foot pool. They swim the length of the pool, back and forth, continuously for ten minutes. If one swimmer swims at a rate of 6 feet per second and the other 4 feet per second, compute the number of times they pass each other. (Assume there is no loss of time when the swimmers turn around.)

PART II *SPRING 2006* *CONTEST 5* *TIME: 10 MINUTES*

S06A27 The centers of two circles are 25 inches apart. The smaller circle has a radius of 2 inches and the larger circle has a radius of 5 inches. Compute the length of the common internal tangent segment.

S06A28 Compute the largest value of x that satisfies the equation: $x^3-19x^2+kx-216=0$, (k is an integer) given that the roots are in geometric progression.

PART III *SPRING 2006* *CONTEST 5* *TIME: 10 MINUTES*

S06A29 Solve for x: $|x^2-5x+6|+|x^2-3x+2|=32$

S06A30 Given that r and s are positive numbers for which: $\log_4 r=\log_6 s=\log_9(2r+3s)$.

Compute, with no logarithms, $\frac{s}{r}$.

ANSWERS:	S06A25	**12,600**
	S06A26	**30**
	S06A27	**24**
	S06A28	**9**
	S06A29	**-2, 6**
	S06A30	$\frac{3+\sqrt{17}}{2}$

New York City Interscholastic Mathematics League

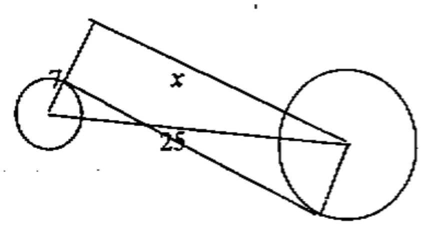
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 5

SPRING 2006 Solutions

S06A25 coefficient $= \binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1} = \frac{10!\,6!\,3!}{6!\,4!\,3!\,3!\,2!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{3\cdot 2\cdot 2} = \mathbf{12,600}$

S06A26 Make a graph with the horizontal axis from x = 0 to x = 100 to represent position in the pool, and with the vertical axis to represent time. The faster swimmer goes from (0, 0) to (100, 50/3) to (0, 100/3) to (100, 50), etc. The slower swimmer, whose rate is 2/3 that of the faster, goes from (100, 0) to (0, 25) to (100, 50), etc. Each oblique segment represents a lap, and the intersection of any two segments represents a moment when the swimmers pass. Continue the above diagram to 100 seconds – 6 laps for the fast swimmer and 4 for the slow one – and note 5 passings. Thus there will be 6*5=30 passings in 600 seconds.



S06A27 If we use the tangent and the radius of the bigger circle as the length and width of a rectangle, then the tangent is the other leg of a right triangle with a leg of 7 and the hypotenuse of 25. Therefore the tangent has a length of **24**.

S06A28 Let the roots $= ar, a, a/r$, then $ar\cdot a\cdot a/r = a^3 = 216 \therefore a = 6$.

$6r + 6 + 6/r = 19 \rightarrow r + 1/r = 13/6 \rightarrow r = 2/3$ or $r = 3/2$.

Thus the roots are 4, 6, 9. The largest root is **9**.

S06A29 $|x^2-5x+6|+|x^2-3x+2|=32$

$x^2-5x+6+x^2-3x+2=32$, $2x^2-8x-24=0 \therefore x=6,-2$ or $x^2-5x+6+-(x^2-3x+2)=32$, $-2x=28 \therefore x=-14$ *reject* or

$-(x^2-5x+6)+x^2-3x+2=32$, $2x=36 \therefore x=18$ *reject*, or $-(x^2-5x+6)-(x^2-3x+2)=32$, $x=2\pm 4i$ *reject*

therefore $x =$ **-2, 6.**

S06A30 Let $a=\log_4 r=\log_6 s=\log_9(2r+3s)$ then $r=4^a \quad s=6^a \quad 9^a=2r+3s$

$9^a=2(4^a)+3(6^a)$ divide both sides by 4^a we get $\left(\frac{9}{4}\right)^a = 2+3\left(\frac{6}{4}\right)^a$

which is equivalent to $\left(\frac{3}{2}\right)^{2a}-3\left(\frac{3}{2}\right)^a-2=0, \quad \therefore \frac{s}{r}=\left(\frac{3}{2}\right)^a=\frac{3+\sqrt{17}}{2}$ (reject the negative answer)