New York City Interscholastic Math League
Senior Division A   Contest Number 1

Part I   Fall 2010   Contest 1   Time: 10 Minutes

F10A1 In a geometric sequence of positive real numbers, the third term is equal to 6 and the twelfth term is equal to $\frac{1}{\sqrt{2}}$. Compute the value of the seventh term.

F10A2 Real numbers u, v, x, y, and z are chosen so that in the diagram at right, the sum of the three entries in each row is equal to 15 and the sum of the three entries in each column is equal to $1+u+z$. Compute x.

1	9	5
u	v	2
z	y	x

Part II   Fall 2010   Contest 1   Time: 10 Minutes

F10A3 In $\triangle ABC$, let M be the midpoint of $\overline{AB}$ and let D be the foot of the altitude from A to $\overline{BC}$. If $AB = 5$, $BC = 7$ and $CA = 9$, compute the length MD.

F10A4 Amanda the ant starts at $(0, 1)$ and travels due north-east at a constant speed of $\sqrt{2}$ units per second. At the same moment, Boateng the beetle starts at $(3, 0)$ and travels due west at a constant speed of 2 units per second. Compute the distance between Amanda and Boateng at the moment when they are as close to each other as they ever get.

Part III   Fall 2010   Contest 1   Time: 10 Minutes

F10A5 Compute the positive integer x that satisfies $\gcd(x, 9900) = 12$ and $\text{lcm}(x, 9900) = 2^3 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11$.

F10A6 A circle passes through the point $(-1, 2)$, is tangent to the x-axis at $(0, 0)$, and is tangent to a line of slope -2 at the point (a, b) with $a > 0$. Compute b.

New York City Interscholastic Math League
Senior Division A   Contest Number 2

Part I   Fall 2010   Contest 2   Time: 10 Minutes

F10A7 How many positive integral divisors of $27,000$ are *not* divisible by 30?

F10A8 The four planes with equations $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 0$ cut space into n regions. Compute n.

Part II   Fall 2010   Contest 2   Time: 10 Minutes

F10A9 In a collection of five integers, the mean is equal to 14, the median is equal to 11 and the unique mode is equal to 9. Compute the largest possible value of the largest member of the collection.

F10A10 Compute the largest positive integer n that *cannot* be written in the form $n = 10a + 11b + 12c$ for nonnegative integers a, b and c.

Part III   Fall 2010   Contest 2   Time: 10 Minutes

F10A11 In $\triangle ABC$, we have that $m\angle C = 90°$, M and N are midpoints of $\overline{BC}$ and $\overline{AC}$, respectively, and G is the intersection of $\overline{AM}$ and $\overline{BN}$. If $AC = 8$ and $BC = 6$, compute CG.

F10A12 There are two ways to fully parenthesize a product of three terms, $(a \times b) \times c$ and $a \times (b \times c)$, and there are five ways to fully parenthesize a product of four terms, $((a\times b)\times c)\times d$, $(a\times(b\times c))\times d$, $(a\times b)\times(c\times d)$, $a \times ((b \times c) \times d)$ and $a \times (b \times (c \times d))$. Compute the number of ways to fully parenthesize a product of seven terms.

New York City Interscholastic Math League
Senior Division A — Contest Number 3

Part I — Fall 2010 — Contest 3 — Time: 10 Minutes

F10A13 Hoda randomly chooses a perfect square larger than 1 and smaller than 50, and Ramis randomly chooses a perfect cube larger than 1 and smaller than 50. Compute the probability that Hoda's chosen number is larger than Ramis's.

F10A14 The product of all positive integral divisors of 360 is equal to $2^a \cdot 3^b \cdot 5^c$ for some integers a, b, and c. Compute the ordered triple (a, b, c).

Part II — Fall 2010 — Contest 3 — Time: 10 Minutes

F10A15 The positive base-7 number $\underline{abc}_7$ is four times larger than the base-3 number $\underline{abc}_3$. Compute the base-10 number $\underline{abc}_{10}$.

F10A16 Function f has the property that for any two positive real numbers x and y, $f(x \cdot y) = f(x) + f(y)$. Function g takes only positive values and has the property that for any real numbers x and y, $g(x+y) = g(x) \cdot g(y)$. If $f(g(1)) = 3$, compute the value $f(g(4))$.

Part III — Fall 2010 — Contest 3 — Time: 10 Minutes

F10A17 Compute the largest positive integer such that its base-10 expansion contains each digit at most once and has no two consecutive digits that differ by exactly one. (For example, 3513 and 8652 do not meet the conditions but 27415 does.)

F10A18 The four points A, B, C, and D are arranged in the plane so that $\frac{m\angle CAD}{m\angle DCA} = \frac{m\angle DCA}{m\angle ADC} = 2$ and $\frac{m\angle BAC}{m\angle BAD} = \frac{m\angle BCA}{m\angle BCD} = 1$. Compute $\frac{m\angle ABC}{m\angle CBD}$.

New York City Interscholastic Math League
Senior Division A   Contest Number 4

Part I Fall 2010 Contest 4 Time: 10 Minutes

F10A19 A cubical box has volume V cubic centimeters and has surface area V square centimeters (including its top, i.e., counting all six faces). Compute, in centimeters, the length of the longest diagonal of the box.

F10A20 Define $\cosh x = \frac{e^x+e^{-x}}{2}$. If $\cosh\alpha = 4$, compute $\cosh 3\alpha$.

Part II Fall 2010 Contest 4 Time: 10 Minutes

F10A21 If $ab+c+d=1$, $b+ac+d=2$, $b+c+ad=3$, $a+b+c+d=4$, and $a>0$, compute a.

F10A22 Compute the smallest positive integer n such that the decimal expansion of $13n$ contains only the digit 1.

Part III Fall 2010 Contest 4 Time: 10 Minutes

F10A23 In $\triangle ABC$, we have $AB=5$, $BC=7$ and $CA=9$. Circle O passes through A and B and also intersects $\overline{AC}$ at M and $\overline{BC}$ at N. If $CN=3$, compute the length MN.

F10A24 To *triangulate* a convex n-gon is to divide its interior into triangles using a collection of diagonals that intersect only at their endpoints. Thus, for example, there are five ways to triangulate a regular pentagon. Compute the number of ways to triangulate a regular octagon.

New York City Interscholastic Math League
Senior Division A Contest Number 5

Part I Fall 2010 Contest 5 Time: 10 Minutes

F10A25 Compute all real values of x for which $4x^2 + 4x^4 + 4x^6 + \ldots = 5$.

F10A26 Let $f(x)$ be a function such that $f(\cos t) = t$ for every $t \in [0, \pi]$. (Here the cosine acts on t as a value given in radians.) Compute $\tan(f(\sin(f(2/5))))$.

Part II Fall 2010 Contest 5 Time: 10 Minutes

F10A27 We have $\{a, b, c, d, e, f\} = \{1, 2, 3, 4, 5, 6\}$. Given that $a+b+c = d$ and $a + e > b + d$, compute f.

F10A28 Compute the smallest positive integer n such that n^2 has exactly three times as many divisors as n.

Part III Fall 2010 Contest 5 Time: 10 Minutes

F10A29 Given that $x + y + z = 7$, $xy + yz + xz = 10$ and $xyz = 5$, compute $(2 - x)(2 - y)(2 - z)$.

F10A30 A point A is selected at random from an $n \times n$ square grid of points. A second point B is selected at random from those points that are *not* in the same row or column as the first point. There exists a unique rectangle that has A and B as opposite vertices and whose sides are parallel to the edges of the grid. Compute, in terms of n, the expected value of the area of this rectangle.

New York City Interscholastic Math League
Senior A Division — Contest Number 1 Solutions

F10A1. $\mathbf{2^{1/3} \cdot 3^{5/9}}$ or $\mathbf{1944^{1/9}}$ or equivalent. Let the common ratio of the sequence be r and let the zeroth term be a. By the givens, $ar^3 = 6$ and $ar^{12} = 2^{-1/2}$. Dividing, $r^9 = 2^{-3/2} \cdot 3^{-1}$ and so $r = 2^{-1/6} \cdot 3^{-1/9}$. Thus the seventh term is $ar^7 = ar^3 \cdot r^4 = 6 \cdot 2^{-2/3} \cdot 3^{-4/9} = 2^{1/3} \cdot 3^{5/9}$.

F10A2. **8**. The sum of the three row-sums and the sum of the three column-sums are both equal to the sum of all nine entries of the square, and so are equal to each other. Since the row-sums are all equal and the column-sums are all equal, the sum of the entries of a single row must equal the sum of the entries of a single column, whence $1+9+5 = 5+2+x$ and $x = 8$. (The given conditions do not allow us to solve for the other four variables; the value of any of them may be chosen arbitrarily, and this choice forces the values of the other three.)

F10A3. $\mathbf{\frac{5}{2}}$. The segment $\overline{MD}$ is the median to the hypotenuse of right triangle $\triangle ABD$, so its length is half the length of the hypotenuse $\overline{AB}$.

F10A4. $\mathbf{\frac{3\sqrt{10}}{5}}$. After t seconds, Amanda is at the point $(t, 1+t)$ and Boateng is at the point $(3-2t, 0)$. Applying the distance formula, we seek to minimize $\sqrt{(3t-3)^2 + (1+t)^2} = \sqrt{10t^2 - 16t + 10}$ for $t \geq 0$. This expression is minimized when $10t^2 - 16t + 10$ is minimized, which occurs at $t = -\frac{-16}{2 \cdot 10} = \frac{4}{5}$. At this moment, the distance between the two is $\sqrt{10 \cdot (4/5)^2 - 15 \cdot (4/5) + 10} = \frac{3\sqrt{10}}{5}$.

F10A5. **1176**. For any positive integers a and b we have $\gcd(a,b) \cdot \operatorname{lcm}(a,b) = ab$. In our case, this implies $x = \frac{12 \cdot 2^3 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11}{9900} = 1176$.

Note that this first argument assumes that the problem is solvable, which is not necessarily the case. (Try to find an integer x such that $\gcd(x, 9) = 3$ and $\operatorname{lcm}(x, 9) = 27$.) An alternative solution without this flaw is to work by prime factorizations. We have $9900 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 11$, so if $x = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e$ we have that $\max(2, a) = 3$ and $\min(2, a) = 2$ (by considering the exponents of 2), that $\max(2, c) = 2$ and $\min(2, c) = 0$ (by considering the exponents of 5), and so on.

F10A6. $\mathbf{\frac{5+\sqrt{5}}{4}}$. Let the length of the radius of the circle be r. Since the circle is tangent to the x-axis at $(0,0)$, its center is $(0, r)$. Since it passes through the point $(-1, 2)$ we have by the distance formula that $r^2 = (-1-0)^2 + (2-r)^2$ and thus $r = \frac{5}{4}$. At any point on a circle, the radius to that point and tangent through that point are perpendicular; since the tangent line to the circle at (a, b) has slope -2, the radius has slope $\frac{1}{2}$. Thus $\frac{b-r}{a} = \frac{1}{2}$, and so $a = 2b - \frac{5}{2}$. We have from the distance formula that $(a-0)^2 + (b-r)^2 = r^2$, so $4b^2 - 10b + 5 = 0$ and $b = \frac{5 \pm \sqrt{5}}{4}$. The smaller value of b is associated with a negative value of a, so the answer is $b = \frac{5+\sqrt{5}}{4}$.

F10A7. **37**. The integer $27000 = 2^3 \cdot 3^3 \cdot 5^3$ has $(3+1)(3+1)(3+1) = 64$ divisors. The number of divisors of 27000 that *are* divisible by 30 is the same as the number of divisors of $\frac{27000}{30} = 2^2 \cdot 3^2 \cdot 5^2$, which is $(2+1)(2+1)(2+1) = 27$. Thus, the number of divisors of 27000 that are *not* divisible by 30 is $64 - 27 = 37$. Alternatively, one can proceed by inclusion-exclusion, counting divisors of 27000 not divisible by 2, not divisible by 3, not divisible by 6, etc., and adding or subtracting as necessary in order to avoid over- or under-counting.

F10A8. **14**. The planes $x = 0$, $y = 0$, and $z = 0$ are the three coordinate planes (the yz-plane, xz-plane and xy-plane, respectively). They divide space into eight octants. When we add an additional plane, it divides each of these eight octants into either one or two pieces – if the new plane passes through the interior of an octant, it cuts it into two pieces, while if it doesn't pass through the interior then it leaves it in one piece.

Each of the eight octants consist of points in which the signs of the coordinates are fixed; for example, $\{(x, y, z) \mid x > 0, y > 0, z > 0\}$ is one octant, while $\{(x, y, z) \mid x > 0, y < 0, z > 0\}$ is another. The plane $x+y+z = 0$ passes through exactly six of the eight octants: it fails to pass through the octants $\{(x, y, z) \mid x > 0, y > 0, z > 0\}$ and $\{(x, y, z) \mid x < 0, y < 0, z < 0\}$. Thus it leaves us with $6 \cdot 2 + 2 = 14$ regions in total.

F10A9. **29**. Since the median element of our collection is different from the modal value, our collection must contain two copies of the value 9, one copy of the value 11, and two distinct integers, say x and y, larger than 11. In addition, these five numbers must sum to 70, so $x + y = 41$. In order to make one of x and y as large as possible, we must make the other as small as possible; this occurs when our collection is $\{9, 9, 11, 12, 29\}$, with largest value 29.

F10A10. **49**. When $a + b + c = 1$, we can get 10, 11 or 12. When $a + b + c = 2$, we can get anything from 20 to 24. When $a + b + c = 3$, we can get anything from 30 to 36. When $a + b + c = 4$, we can get anything from 40 to 48. When $a + b + c = 5$, we can get anything from 50 to 60. From this point onwards, the ranges overlap and so we can get every value larger than 60. The largest value we didn't get is 49.

Challenge problem: what if we replace 10, 11 and 12 with $2m$, $2m + 1$ and $2m + 2$ for some positive integer m?

F10A11. $\mathbf{\frac{10}{3}}$. The point G is the centroid of $\triangle ABC$, so its distance to C is two-thirds of the length of the median from C. The median to the hypotenuse of a right triangle has length equal to half the hypotenuse, which in this case is of length 10. Thus $CG = \frac{2}{3} \cdot \frac{1}{2} \cdot 10 = \frac{10}{3}$.

F10A12. **132**. Let P_n denote the number of full parenthesizations (henceforth FPs) of a product of n terms. Thus, we have $P_1 = P_2 = 1$ (using no parentheses), $P_3 = 2$ and $P_4 = 5$. To compute P_n, observe that in an FP of two or more terms, there is always a single multiplication sign not enclosed inside any matched pair of parentheses. This multiplication sign splits the product into two factors, each of which is itself an FP. For example, in $(a \times (b \times c)) \times ((d \times e) \times f)$, the third multiplication sign separates the two factors $a \times (b \times c)$ and $(d \times e) \times f$, each of which is an FP of three terms. If this special multiplication sign is the kth one from the left, there are k terms to its left and $n - k$ to its right, so $P_k \cdot P_{n-k}$ possible

FPs. Thus we have in total $P_n = P_1 \cdot P_{n-1} + P_2 \cdot P_{n-2} + \ldots + P_{n-2} \cdot P_2 + P_{n-1} \cdot P_1$ FPs of a product of n terms. Applying this for $n = 5, 6$ and 7 gives $P_5 = P_1 \cdot P_4 + P_2 \cdot P_3 + P_3 \cdot P_2 + P_4 \cdot P_1 = 5 + 2 + 2 + 5 = 14$, $P_6 = 14 + 5 + 4 + 5 + 14 = 42$ and $P_7 = 42 + 14 + 10 + 10 + 14 + 42 = 132$.

The numbers P_n are called *Catalan numbers* after the 19th century mathematician Eugène Catalan. They have the simple formula $P_n = \frac{1}{n}\binom{2n-2}{n-1}$ (a fact which is rather complicated to prove even with the recursive definition derived above) and appear in an astounding array of different combinatorial contexts; the Wikipedia article on Catalan numbers contains a good introduction to some of these different contexts.

New York City Interscholastic Math League
Senior A Division — Contest Number 3 Solutions

F10A13. $\mathbf{\frac{7}{12}}$. Hoda chooses from among $\{4, 9, 16, 25, 36, 49\}$ while Ramis chooses from among $\{8, 27\}$. If Ramis chooses 8, the probability that Hoda chooses a larger value is $\frac{5}{6}$ (as any value other than 4 will do). If Ramis chooses 27, the probability that Hoda chooses a larger value is $\frac{1}{3}$. Since Ramis's two choices are equally likely, the probability we desire is $\frac{1}{2} \cdot \frac{5}{6} + \frac{1}{2} \cdot \frac{1}{3} = \frac{7}{12}$.

F10A14. $(\mathbf{36}, \mathbf{24}, \mathbf{12})$. We have $360 = 2^3 \cdot 3^2 \cdot 5$, so 360 has $(3+1)(2+1)(1+1) = 24$ divisors. Of these, half are not divisible by 5 while half are divisible by 5 but not 5^2; one third are not divisible by 3, one third are divisible by 3 but not by 3^2, and one third are divisible by 3^2 but not by 3^3; one quarter are odd, one quarter are divisible by 2 but not 2^2, one quarter are divisible by 2^2 but not 2^3 and one quarter are divisible by 2^3 but not 2^4. Thus, the total power of 5 in our product is $12 \cdot 0 + 12 \cdot 1 = 12$, the total power of 3 is $8 \cdot 0 + 8 \cdot 1 + 8 \cdot 2 = 24$ and the total power of 2 is $6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 = 36$.

Alternatively, let the 24 factors of 360 be $f_1 < f_2 < \ldots < f_{24}$. Then $f_1 \cdot f_2 \cdots f_{24} = (f_1 \cdot f_{24}) \cdot (f_2 \cdot f_{23}) \cdots = 360 \cdot 360 \cdots = 360^{12}$. Now it's easy to finish.

Challenge: generalize! What if we replace 360 by the number with prime factorization $p_1^{a_1} \cdot p_2^{a_2} \cdot \cdots \cdot p_k^{a_k}$?

F10A15. **121**. We have $49a + 7b + c = 4(9a + 3b + c)$, so collecting terms gives $13a = 5b + 3c$. Moreover, we must have that $a > 0$ and that each of a, b, c is a valid digit in base 3, i.e., $0 \le a, b, c \le 2$. An exhaustive search quickly shows that the only solution is $a = 1$, $b = 2$, $c = 1$, leading to the answer 121.

F10A16. **12**. We have $f(g(4)) = f(g(2+2)) = f(g(2) \cdot g(2)) = f(g(2)) + f(g(2)) = 2f(g(2))$. Similarly, $f(g(2)) = 2f(g(1)) = 6$. Thus $f(g(4)) = 12$.

Another approach is to try to find two functions f and g with the desired properties, on the assumption (implicit in the question) that the answer is independent of this choice. The property that defines the function f is characteristic of logarithmic functions, while the property defining the function g is characteristic of exponential functions; thus, a natural choice might be $g(x) = 8^x$ and $f(x) = \log_2(x)$, or similar. Note that exponential and logarithmic functions are not the only functions satisfying these conditions; however, they are the only "nice" such functions. (For example, the only *continuous* functions with the given properties are logarithms and exponentials.)

F10A17. **9758642031**. If we build such a number, we should start at the leftmost digit and always place the largest available digit, provided that we can complete this procedure using all ten digits. Thus, the leftmost digit should be 9. Given that choice, the next digit should be 7 (because we can't place 8 next to 9) and the third should be 5 (because we can't place 8 or 6 next to 7). Then the fourth digit from the left should be 8, then 6, then 4, then 2 (because we can't place 3 next to 4), then 0 (because we can't place 3 or 1 next to 2), then 3, then 1. Since we have managed to include all ten digits, the resulting number 9758642031 is the largest possible.

Challenge problem: what about other bases? In particular, notice that things are more complicated in bases 2, 3, 4 and 6, while they are straightforward in bases 5, 7 and 10 – can you establish which bases are "nice" and which are complicated?

F10A18. $\mathbf{\frac{8}{11}}$. From the second set of equations we have $\angle BAC \cong \angle BAD$ and $\angle BCA \cong \angle BCD$. The first congruence implies that either A, C and D are colinear or that $\overleftrightarrow{AB}$ is the angle bisector of $\angle CAD$. Similarly, the second congruence implies that either A, C and D are colinear or that $\overleftrightarrow{BC}$ is the angle bisector of $\angle ACD$. From the first set of equations, we see that A, C and D can not be colinear (we would have either division by 0 or $0 = 2$), so B is the intersection of the angle bisectors of $\angle A$ and $\angle C$ in $\triangle ACD$. Thus B is the incenter of $\triangle ACD$ and also lies on the angle bisector of $\angle D$.

Now from the first set of equations we have that $m\angle CAD = 2m\angle DCA = 4m\angle ADC$. Since $m\angle CAD + m\angle DCA + m\angle ADC = \pi$, we have $m\angle ADC = \frac{\pi}{7}$, $m\angle DCA = \frac{2\pi}{7}$ and $m\angle CAD = \frac{4\pi}{7}$. By the preceding paragraph, we are left with a quick angle-chase; we find $m\angle ABC = \frac{4\pi}{7}$ and $m\angle CBD = \frac{11\pi}{14}$ and so their ratio is $\frac{8}{11}$.

New York City Interscholastic Math League
Senior A Division — Contest Number 4 Solutions

F10A19. $\mathbf{6\sqrt{3}}$. Let the side-length of the box be s. Then we have $6s^2 = s^3$ and so $s = 6$. Thus the long diagonal of the box has length $\sqrt{6^2+6^2+6^2} = 6\sqrt{3}$.

F10A20. **244**. We have $\cosh(3x) = \frac{e^{3x}+e^{-3x}}{2}$ and $\cosh^3 x = \left(\frac{e^x+e^{-x}}{2}\right)^3 = \frac{e^{3x}+3e^x+3e^{-x}+e^{-3x}}{8}$. Thus $\cosh(3x) = 4\cosh^3 x - 3\cosh x$. Setting $x = \alpha$ implies $\cosh 3\alpha = 4 \cdot 4^3 - 3 \cdot 4 = 244$. (This is the hyperbolic trig analogue of the triple-angle formula for cosine. Most trig identities (including those that arise in elementary trigonometry, such as the Pythagorean identity and the angle sum formulas, and those that arise in calculus) have corresponding hyperbolic trig identities – often the only difference between the "usual" and hyperbolic identities is some change of sign. This is closely related to Euler's identity $e^{ix} = \cos x + i \sin x$.)

F10A21. $\mathbf{1+\sqrt{3}}$. Add the first three equations to get $(a+2)(b+c+d) = 6$. Let $q = b+c+d$, so we have $(a+2)q = 6$ and $a+q = 4$. Now solve as you like; for example, eliminate q by substituting from the second equation into the first to get $(a+2)(4-a) = 6$, so $a^2 - 2a - 2 = 0$ or $a = 1 \pm \sqrt{3}$. We were asked only for the positive solution, which is $1+\sqrt{3}$.

F10A22. **8547**. If an integer has all of its digits equal to 1, we call it a *repunit*. Since $13n$ is a repunit it has units digit 1, so n must have units digit 7, i.e., $n = 10a+7$. Thus $13n = 130a + 91$, and so $13a + 9 = \frac{13n-1}{10}$ is also a repunit. In particular, $13a+9$ has units digit 1, so a must have units digit 4, i.e., $a = 10b+4$. Thus $13b + 6 = \frac{(13a+9)-1}{10}$ is again a repunit and so has units digit 1. Therefore the units digit of b must be equal to 5, so $b = 10c+5$ and consequently $13c + 7 = \frac{(13b+6)-1}{10}$ is a repunit. Thus c must have units digit 8 and so $c = 10d+8$, whence $13d + 11 = \frac{(13c+6)-1}{10}$ is a repunit. Now we see that we can take $d = 0$, giving $c = 8$, $b = 85$, $a = 854$ and $n = 8547$ as the smallest such number.

Alternatively, note that the k-digit repunit is equal to $\frac{10^k-1}{9}$. For this to be divisible by 13, we must have $13 | 10^k - 1$ or $10^k \equiv 1 \pmod{13}$. Fermat's Little Theorem says that for a prime p and an integer a not divisible by p, $a^{p-1} \equiv 1 \pmod{p}$, so $k = 12$ is a solution to the congruence in the previous sentence; however, FLT does not guarantee that it is the *smallest* solution. We must check whether any divisors of 12 also yield a solution; it's not hard to confirm by dividing that for $k = 6$, we have $111111 = 13 \cdot 8547$, while no other shorter repunit is divisible by 13.

F10A23. $\mathbf{\frac{5}{3}}$. Either use power-of-a-point to find $CM = \frac{7}{3}$ to recognize an SAS similarity between $\triangle NCM$ and $\triangle ABC$, or use the properties of cyclic quadrilaterals to get $m\angle BAC = 180^\circ - m\angle BNM = m\angle CNM$ and so recognize an AA similarity. In either case, we have $\triangle ABC \sim \triangle NCM$ with ratio of similarity $\frac{1}{3}$, so $MN = \frac{AB}{3} = \frac{5}{3}$.

F10A24. **132**. Let T_n be the number of triangulations of an n-gon, and for convenience define $T_2 = 1$. It's easy to see that $T_3 = 1$ (there are no diagonals to draw), $T_4 = 2$ (take a single diagonal) and $T_5 = 5$ (mentioned in the problem; there are five pairs of noncrossing diagonals dividing the pentagon into three triangles). In any triangulation of the $(n+1)$-gon $P = A_1A_2\cdots A_{n+1}$, there exists a triangle $\triangle A_{n+1}A_1A_i$ for some $i \in \{2, \ldots, n\}$, i.e., with $\overline{A_{n+1}A_1}$ as one of its edges. This triangle divides P into two pieces, a "right piece"

$A_1A_2\cdots A_i$ with i vertices and a "left piece" $A_iA_{i+1}\cdots A_{n+1}$ with $n+2-i$ vertices. To complete the triangulation of P, we need to choose a triangulation for each of these pieces, which can be done in $T_i \cdot T_{n+2-i}$ ways. It follows that $T_{n+1} = T_2 \cdot T_n + T_3 \cdot T_{n-1} + \ldots + T_n \cdot T_2$. We can use this recursion to quickly compute $T_6 = 14$, $T_7 = 42$ and $T_8 = 132$. This is another manifestation of the Catalan numbers – see the solution to question 12 for more information.

New York City Interscholastic Math League

Senior A Division   Contest Number 5 Solutions

F10A25. $\frac{\sqrt{5}}{3}, -\frac{\sqrt{5}}{3}$. (Both answers are required!) We have an infinite geometric series with first term $4x^2$ and common ratio x^2. Such a series has sum $\frac{4x^2}{1-x^2}$, so $4x^2 = 5(1-x^2)$ and $x^2 = \frac{5}{9}$. It follows that the two solutions are $x = \frac{\sqrt{5}}{3}$ and $x = -\frac{\sqrt{5}}{3}$.

F10A26. $\frac{2\sqrt{21}}{21}$. By definition, $f(2/5)$ is the measure of the angle whose cosine is $2/5$. This angle is the acute angle adjacent to the leg of length 2 in a right triangle with hypotenuse 5 and legs 2 and $\sqrt{5^2-2^2} = \sqrt{21}$. Thus $\sin(f(2/5)) = \frac{\sqrt{21}}{5}$. It follows that $f(\sin(f(2/5)))$ is the other acute angle in the same triangle, and so that $\tan(f(\sin(f(2/5)))) = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$.

F10A27. **4**. Since a, b and c are distinct positive integers we have $a+b+c \geq 1+2+3 = 6$, while $d \leq 6$. Thus, the only way we can have $a+b+c = d$ is if both inequalities are actually equalities, i.e., if $\{a,b,c\} = \{1,2,3\}$ and $d = 6$. Thus also $\{e,f\} = \{4,5\}$. Now note that $a+e \leq 3+5 = 8$ while $b+d \geq 1+6 = 7$, so again the only way we can have $a+e > b+d$ is if both inequalities are equalities. Thus $e = 5$ and so $f = 4$.

F10A28. **144**. Write $n = p_1^{a_1} \cdot p_2^{a_2} \cdots$ for primes $p_1, p_2, \ldots$ and positive integers $a_1, a_2, \ldots$. Then the number of divisors of n is $(a_1+1)\cdot(a_2+1)\cdots$ and the number of divisors of n^2 is $(2a_1+1)\cdot(2a_2+1)\cdots$. Note that $\frac{3}{2} \leq \frac{2x+1}{x+1} < 2$ for every positive integer x. Since $\left(\frac{3}{2}\right)^3 = \frac{27}{8} > 3$ we have that if an integer has three or more prime factors, its square will have more than three times as many factors. Thus, n has at most two prime factors. The equation $2a_1+1 = 3(a_1+1)$ has no positive integer solutions, so actually n has exactly two prime factors and we must solve $(2a_1+1)(2a_2+1) = 3(a_1+1)(a_2+1)$ for positive integers a_1, a_2. Since the left side of this equation is odd, a_1 and a_2 must both be even. If $a_1 = a_2 = 2$ the right-hand side is larger than the left-hand side; $\{a_1, a_2\} = \{2, 4\}$ is a solution; and for any other pair (a_1, a_2) we have that the left-hand side is larger than the right-hand side. Thus $n = p_1^4 \cdot p_2^2$. The smallest such n is $2^4 \cdot 3^2 = 144$.

Alternatively, to solve $(2a_1+1)(2a_2+1) = 3(a_1+1)(a_2+1)$ we may expand and collect terms. The equation becomes $a_1a_2 - a_1 - a_2 = 2$ or $(a_1-1)(a_2-1) = 3$. Since 3 is prime, the only positive integral solution is $\{a_1, a_2\} = \{2, 4\}$.

F10A29. $-\mathbf{5}$. Expanding out, we have $(2-x)(2-y)(2-z) = 8 - 4(x+y+z) + 2(xy + yz + xz) - xyz = 8 - 4\cdot 7 + 2\cdot 10 - 5 = -5$.

F10A30. $\frac{(n+1)^2}{9}$. There are $n^2 \cdot (n-1)^2$ ways to choose the two points. Each rectangle is chosen by this process in exactly four ways, so our answer is four times the total area of all (axis-parallel) rectangles in the grid divided by $n^2(n-1)^2$. Fix i between 1 and $n-1$, and let us count the total area of all rectangles of width i. There are $n-i$ pairs of vertical grid lines forming the left and right edges of a rectangle of width i. Once we've fixed the left and right edges, there are $n-1$ rectangles of height 1, $n-2$ rectangles of height 2, …, and one rectangle of height $n-1$. Thus, the total area of these rectangles is $(n-i)\sum_{j=1}^{n-1} j \cdot (i \cdot (n-j))$.

We may rewrite this as $(n-i)i\sum_{j=1}^{n-1}(n-j)j$. We may compute

$$\begin{aligned}\sum_{j=1}^{n-1}(n-j)j &= n\left(\sum_{j=1}^{n-1} j\right) - \sum_{j=1}^{n-1} j^2 \\ &= \frac{n^2(n-1)}{2} - \frac{(n-1)n(2n-1)}{6} \\ &= \frac{(n+1)n(n-1)}{6},\end{aligned}$$

so the total area of all the rectangles in the grid of width i is $(n-i)i \cdot \frac{(n+1)n(n-1)}{6}$. It follows that the total area of all rectangles in the grid is

$$\sum_{i=1}^{n-1}(n-i)i \cdot \frac{(n+1)n(n-1)}{6} = \frac{(n+1)^2 n^2 (n-1)^2}{36}.$$

Finally, to compute the desired expected value we multiply by 4 and divide by $n^2(n-1)^2$ to get $\frac{(n+1)^2}{9}$.