NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 1

PART I *FALL 2006* *CONTEST 1* *TIME: 10 MINUTES*

F06A1 Compute
$$\left(1-\sqrt{6}\right)\left(1-\sqrt{5}\right)\left(1-\sqrt{4}\right)\left(1-\sqrt{3}\right)\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)\left(1+\sqrt{3}\right)\left(1+\sqrt{4}\right)\left(1+\sqrt{5}\right)\left(1+\sqrt{6}\right).$$

F06A2 A triangle has sides of length 10, 24, and 26. Compute the distance between the center of its inscribed circle and the center of its circumscribed circle.

PART II *FALL 2006* *CONTEST 1* *TIME: 10 MINUTES*

F06A3 Mario, Luigi and Peach take turns driving. Each drives one-third the total distance of the trip. If Mario drives 40 mph, Luigi 50 mph and Peach 60 mph, compute their average speed for the entire trip in mph.

F06A4 The sum of two numbers is 2. The sum of their cubes is 152. Compute the sum of their squares.

PART III *FALL 2006* *CONTEST 1* *TIME: 10 MINUTES*

F06A5 An m – sided regular polygon has an $n°$ interior angle. If the number of sides were doubled, the interior angles would be 18° greater. Compute $m + n$.

F06A6 Compute the probability that a positive four-digit integer containing only the digits 1, 2, 3, or 4 (digits may be repeated) is divisible by 3.

ANSWERS:

F06A1	**-120**
F06A2	$\sqrt{65}$
F06A3	$\frac{1800}{37}$ or $48\frac{24}{37}$
F06A4	**52**
F06A5	**154**
F06A6	$\frac{85}{256}$

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F06A1   **-120.** Multiply the conjugates $(1-\sqrt{6})(1+\sqrt{6})=1-6=-5$ and so on. The product simplifies to $(-5)(-4)(-3)(-2)(-1)=-120$.

F06A2 $\sqrt{65}$. The triangle is a right triangle. Place the right angle at the origin, with one vertex at (24, 0) and the other at (0, 10). The center of the circumscribed rectangle is the midpoint of the hypotenuse (12, 5). Use the fact that tangents to a circle from a common exterior point are congruent to find the radius of the inscribed circle to be 4. Therefore the center of the inscribed circle is at (4, 4). The distance from (4, 4) to (12, 5) is $\sqrt{(12-4)^2+(5-4)^2}=\sqrt{65}$.

F06A3 $\mathbf{\frac{1800}{37}}$. Let d = the distance each travels then the total distance traveled is $3d$. The time that each drives may be calculated using $\frac{\text{distance}}{rate}=time$. Mario drives for $\frac{d}{40}hours$, Luigi drives $\frac{d}{50}hours$ and Peach drives for $\frac{d}{60}hours$. The total time traveled is $\frac{d}{40}+\frac{d}{50}+\frac{d}{60}=\frac{74d}{1200}=\frac{37d}{600}$. The average rate = $\frac{3d}{\left(\frac{37d}{600}\right)}=\frac{1800}{37}mph$ or $48\frac{24}{37}$ mph

F06A4   **52.** We are given $x+y=2$ and $x^3+y^3=152$. Cubing the first equation we get: $(x+y)^3=x^3+y^3+3x^2y+3xy^2=8$. $\therefore 8=152+3xy(x+y)$ or $-144=3xy(2)$.
So $xy=-24$. Since $(x+y)^2=x^2+y^2+2xy \rightarrow 2^2=x^2+y^2-48 \rightarrow x^2+y^2=52$.

F06A5   **154.** The exterior angle of an m – sided regular polygon has a measure $=\frac{360^\circ}{m}$.
Therefore $\frac{360^\circ}{m}=180^\circ-n$ and $\frac{360^\circ}{2m}=180^\circ-(n+18^\circ)$. Subtract the two equations to get $\frac{360^\circ}{2m}=18^\circ \rightarrow m=10^\circ$. Substitute $m=10$ in the first equation and $n=144$ and $m+n=154$.

F06A6   **85/256**   Let S be the sum of the digits.. Because $4\le S\le 16$ and S must be a multiple of 3, it follows that S=6, 9, 12 or 15. Let the digits be a, b, c, and d. Then when S = 6, $\{a, b, c, d\}$ = {1, 1, 1, 3} or {1, 1, 2, 2}. In the first case, there are 4 four-digit numbers with those digits, and 6 such numbers in the second case, a total of 10. Similarly, when S = 9, {a, b, c, d} = {1, 1, 3, 4}, {1, 2, 2, 4}, {1, 2, 3, 3}, or {2, 2, 2, 3}; the number of four-digit numbers are 12, 12, 12, and 4, respectively, a total of 40. When S = 12, $\{a, b, c, d\}$ = {1, 3, 4, 4}, {2, 2, 4, 4}, {2, 3, 3, 4}, or {3, 3, 3, 3}; the number of four-digit numbers are 12, 6, 12, and 1, respectively, a total of 31. Finally, when S = 15, $\{a, b, c, d\}$ = {3, 4, 4, 4}, and the number of four-digit numbers is 4. Thus there are 10 + 40 + 31 + 4 = 85 numbers with the requested property. Because there are $4^4=256$ four-digit numbers with the digits 1, 2, 3, or 4, the probability is 85/256.
OR: The problem is equivalent to finding the number of arrangements of 1, 2, 3, and 4 whose sum is a multiple of 3. Consider the generating polynomial $(x+x^2+x^3+x^4)^4$. For each n, the coefficient of x^n in its expansion is the number of arrangements of 1, 2, 3, and 4 whose sum is n. Now

$$(x+x^2+x^3+x^4)^4=(x^2+2x^3+3x^4+4x^5+3x^6+2x^7+x^8)^2$$
$$=x^4+4x^5+10x^6+20x^7+31x^8+40x^9+44x^{10}+40x^{11}+31x^{12}+20x^{13}+10x^{14}+4x^{15}+x^{16}$$

so the desired number of arrangements is the sum of the coefficients of the terms whose exponents are multiples of 3, namely, 10 + 40 + 31 + 4 = 85. Because there are 4^4 = 256 four-digit numbers with the digits 1, 2, 3, or 4, the probability is 85/256.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 2

PART I *FALL 2006* *CONTEST 2* *TIME: 10 MINUTES*

F06A7 If k and n are integers and $(3^{2006}+2006)^2-(3^{2006}-2006)^2=k\cdot 3^n$, where k is not divisible by 3, compute $\frac{n+k}{2006}$.

F06A8 If $\sqrt{43-15\sqrt{8}}=a+b\sqrt{c}$, where a, b, and c are integers, a is greater than 0 and c is not divisible by the square of any prime, compute $a+b+c$.

PART II *FALL 2006* *CONTEST 2* *TIME: 10 MINUTES*

F06A9 Two fair six-sided dice are rolled over and over. Compute the probability that a seven will appear before an eleven appears.

F06A10 When the expression $(x+2y+3z)^9$ is expanded, the coefficient of the term $x^3y^3z^3$ is n. Compute $\frac{n}{9!}$.

PART III *FALL 2006* *CONTEST 2* *TIME: 10 MINUTES*

F06A11 On a game show, the letters C, A, and R (10 of each) are randomly hidden behind the numbers from 1 to 30 (one letter per number). If a contestant chooses three different numbers, compute the probability that the letters chosen can be used to spell the word CAR.

F06A12 ABC is an equilateral triangle with side length of 15. D is on $\overline{BC}$ such that $BD=\frac{1}{3}BC$. E is on $\overline{AB}$ such that when we construct $\overline{ED}$, $AE=ED$. Compute CE.

ANSWERS:

F06A7	**5**
F06A8	**4**
F06A9	$\frac{3}{4}$
F06A10	**1**
F06A11	$\frac{50}{203}$
F06A12	**13**

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F06A7 **5.** Let $a=3^{2006}$ and $b=2006$. $(a+b)^2-(a-b)^2=4ab$.
$4\cdot 3^{2006}\cdot 2006=k\cdot 3^n \rightarrow k=4\cdot 2006$. (note 8024 is not divisible by 3). $\frac{n+k}{2006}=\frac{4\cdot 2006+2006}{2006}=$ **5.**

F06A8 **4.** $\sqrt{43-15\sqrt{8}}=\sqrt{43-30\sqrt{2}}=a+b\sqrt{c}$ Squaring both sides, we get $43-30\sqrt{2}=a^2+b^2c+2ab\sqrt{c}$ or $a^2+b^2c=43$ and $ab=-15$ and $c=2$. Since a, b, and c are integers, $a=1,3,5,15$. By trial and error, we find $a=5$. $\therefore b=-3$ and $a+b+c=4$.

OR: Let $\sqrt{43-15\sqrt{8}}=\sqrt{x}-\sqrt{y}$. Square both sides to get $43-15\sqrt{8}=x+y-2\sqrt{xy}$. Let $x+y=43$ and $2\sqrt{xy}=15\sqrt{8}$. Then $xy=450=18\bullet 25$. Thus $\sqrt{43-15\sqrt{8}}=\sqrt{25}-\sqrt{18}=5-3\sqrt{2}$.
$a=5$. $\therefore b=-3$ and $a+b+c=4$

F06A9 $\frac{3}{4}$ The answer can be obtained by looking at P(7 on the 1st roll) + P(7 on the 2nd roll before an 11) + P(7 on the 3rd roll before an 11) + … = P(7) + P(not 7, not 11)P(7) + P(not 7, not 11)P(not 7, not 11)P(7) + … = $(1/6)+(7/9)(1/6)+(7/9)^2(1/6)+\ldots=\frac{3}{4}$.

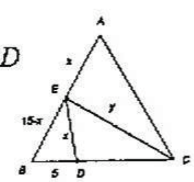
OR: Consider the experiment consisting of rolling two dice and noting the total. The probability of rolling a 7 is 1/6 and the probability of rolling an 11 is 1/18, so rolling a 7 is 3 times as likely as rolling an 11. Now consider the experiment consisting of rolling two dice and noting only totals of 7 and 11, ignoring all other totals. The sample space consists of only 7's and 11's, with 7 being 3 times as probable as 11. Thus the probability of a 7 being the first roll noted is $\frac{3}{4}$.

F06A10 **1.** To compute the coefficient for $x^3y^3z^3$, we need to choose 3 x's out of 9 letters then 3 y's out of 6 and then $3z$'s out of 3. ${}_9C_3\,{}_6C_3\,{}_3C_3(x)^3(2y)^3(3z)^3=\frac{9!}{6!3!}\frac{6!}{3!3!}\frac{3!}{3!0!}2^33^3x^3y^3z^3$

$n=\frac{9!}{3!3!3!}2^33^3=9!\rightarrow\frac{n}{9!}=$ **1.**

F06A11 $\mathbf{\frac{50}{203}}$. The first letter could be C, A or R. The next letter must not match the first. There are 20 good letters out of the remaining 29. The third must not match the first two. There are 10 good letters out of 28 remaining letters. $P(\text{spelling out CAR})=1\left(\frac{20}{29}\right)\left(\frac{10}{28}\right)=\frac{50}{203}$

F06A12 **13.** Let $AE=ED=x\rightarrow BE=15-x$. Use the law of cosines in triangle EBD to get: $x^2=5^2+(15-x)^2-2\cdot 5(15-x)\cos 60^\circ=175-25x+x^2\rightarrow 0=175-25x\rightarrow x=7$.
Let $CE=y$ and use the law of cosines in triangle AEC to get:
$y^2=7^2+15^2-2\cdot 7\cdot 15\cdot\cos 60^\circ=169\rightarrow y=13$.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 3

PART I *FALL 2006* *CONTEST 3* *TIME: 10 MINUTES*

F06A13 Point $T(3,5)$ and point $M(-5, 6)$ are opposite vertices of square *TIME*. Point I has coordinates (a, b) and $E(c, d)$. I and E are the other two vertices of the square. Compute $a + b + c + d$.

F06A14 To determine his student's grades, Mr. Escalante rolls two-ten-sided dice with the digits from 0 to 9 on the sides. He uses the higher roll for the ten's digit and the smaller roll as the unit's digit. If the rolls are equal, he uses the rolls as the ten's and unit's digit. Compute the probability that the student scores at least a 65 on a single roll.

PART II *FALL 2006* *CONTEST 3* *TIME: 10 MINUTES*

F06A15 Given that $n \cdot \frac{10!}{5!} = k!$, where n and k are positive integers. Compute the minimum value of n.

F06A16 If $\frac{7}{2007} < \frac{a}{a+b} < \frac{8}{2007}$, compute the number of integral values for $\frac{b}{a}$.

PART III *FALL 2006* *CONTEST 3* *TIME: 10 MINUTES*

F06A17 Kevin has 2006 marbles (1003 red and 1003 blue). He divides the marbles among two jars. One jar has only red marbles and each jar contains at least 6 marbles. He places the marbles such that if a jar is chosen at random and a marble is selected from that jar, the probability that the marble is red is a maximum. Compute the probability that a red marble is chosen.

F06A18 There are exactly six integers n such that $\frac{(n+9)^2}{n+26}$ is an integer. Compute the largest value of $|n|$.

ANSWERS:		
	F06A13	**9**
	F06A14	$\frac{27}{50}$
	F06A15	**12**
	F06A16	**36**
	F06A17	$\frac{2997}{4000}$
	F06A18	**315**

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F06A13 **9.** Both diagonals have the same midpoint. To find the midpoint, we take the average of the two vertices. The midpoint of $TM = \left(\frac{3-5}{2}, \frac{5+6}{2}\right) = (-1, 5.5)$. Therefore

$\frac{a+c}{2} = -1 \to a+c = -2, \frac{b+d}{2} = 5.5 \to b+d = 11$. Now $a + b + c + d = 9$.

F06A14 $\frac{27}{50}$. We can compute the probability a student gets at least a 65 by computing the probability of scoring a 66 or less and subtracting the probability of scoring a 66 or 65. To score 66 or less, both dice rolls must be 6 or less. $P(\text{both dice are 6 or less}) = \frac{7}{10} \bullet \frac{7}{10} = \frac{49}{100}$ and

$P(\text{scoring a 66 or 65}) = \frac{3}{100}$. $P(\text{scoring at least a 65}) = 1 - \frac{46}{100} = \frac{54}{100} = \frac{27}{50}$.

F06A15 **12.** $n \bullet \frac{10 \bullet 9 \bullet 8 \bullet 7 \bullet 6 \bullet 5!}{5!} = k!$ or $n \bullet 10 \bullet 9 \bullet 8 \bullet 7 \bullet 6 = k!$ To minimize n, we need to minimize k.

Factor the left side of the equation, $n \bullet 5 \bullet 2 \bullet 3 \bullet 3 \bullet 4 \bullet 2 \bullet 7 \bullet 6 = k!$ or $n \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1 \bullet 3 \bullet 2 = k!$ Because there is an extra factor of 2 and 3 after 7!, $k!$ must have an 8 and a 9. So the left side of the equation is missing a factor of 4 and 3. $n = 12$.

F06A16 **36.** The given inequalities imply that $\frac{2007}{7} > \frac{a+b}{a} > \frac{2007}{8}$. Then $286\frac{5}{7} > 1 + \frac{b}{a} > 250\frac{7}{8}$,

so $285 \geq \frac{b}{a} \geq 250$. Hence there are 36 integral values for $\frac{b}{a}$.

F06A17 $\frac{2997}{4000}$. Since there must be at least 6 marbles in each jar, place only 6 red marbles in the first jar and the rest in the other jar with the blue marbles. This will give the red marbles a 100% chance of being selected if the first jar is chosen. $P(red) = \frac{1}{2}(1) + \frac{1}{2}\left(\frac{997}{2000}\right) = \frac{2997}{4000}$. (Note moving one red from jar #2 to jar #1 changes the probability for the worst, because it keeps the same the P(red in the first jar) but decreases the P(red in the second jar) i.e.

$P(red) = \frac{1}{2}\left(\frac{7}{7}\right) + \frac{1}{2}\left(\frac{996}{1999}\right) = \frac{7 \bullet 1999 + 7 \bullet 996}{2 \bullet 7 \bullet 1999} = \frac{20965}{27986}$)

OR: Let r be the number of red marbles in the second jar. Because this jar contains 1003 blue marbles, it has a total of 1003 + r marbles. The probability p of choosing a red marble is given by $p = \frac{1}{2} \bullet 1 + \frac{1}{2} \bullet \frac{r}{1003+r}$, which is maximum when

$\frac{1003+r}{r}$ is minimum. Because $\frac{1003+r}{r} = 1 + \frac{1003}{r}$, it follows that p is maximum when r is maximum, namely when $r = 997$.

F06A18 **315.**

$\frac{(n+9)^2}{n+26} = \frac{n^2+18n+81}{n+26} = \frac{n^2+18n-208+289}{n+26} = \frac{(n-8)(n+26)+289}{n+26} = n - 8 + \frac{289}{n+26}$. Thus $n + 26$ divides 289 evenly. The factors of 289 are $\pm1, \pm17, \pm289$. $n + 26 = \pm1, \pm17, \pm289$ The maximum n = 263 and the minimum n= -315. The maximum $|n| = 315$.

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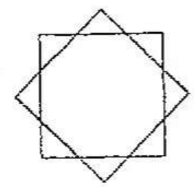
PART I   *FALL 2006*   *CONTEST 4*   *TIME: 10 MINUTES*

F06A19   If $(\log_b a)(\log_c b)(\log_c a) = 25$ and $\frac{a^2}{c^2} = c^k$, compute the sum of all possible values of k.

F06A20   Two concentric congruent squares overlap to form a new16-sided polygon. All the sides of the new polygon are congruent. The ratio of the area of the sixteen-sided polygon to the area of one of the squares is represented as $a + b\sqrt{2}$. If a and b are integers, compute $a + b$.



PART II   *FALL 2006*   *CONTEST 4*   *TIME: 10 MINUTES*

F06A21   The natural number n is written in decimal notation. If the digit 2 is added to the right, the resulting number is a multiple of 9. When this new number is divided by 9, the quotient is 21 more than n. Compute n.

F06A22   Compute $(\sin 165^\circ - \sin 75^\circ)^8$.

PART III   *FALL 2006*   *CONTEST 4*   *TIME: 10 MINUTES*

F06A23   Compute the number of ordered pairs (x, y) of positive integers that satisfy $x^2 - y^2 = 2006$.

F06A24   In triangle ABC, $AB = 20$, $BC = 13$ and $AC = 21$. Given $\cos A + \cos B + \cos C = \frac{p}{q}$, where p and q are relatively prime. Compute $p + q$.

ANSWERS:

F06A19	**-4**
F06A20	**2**
F06A21	**187**
F06A22	$\mathbf{\frac{1}{16}}$
F06A23	**0**
F06A24	**158**

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F06A19 **-4.** $\log_b(a)\cdot\log_c(b)\cdot\log_c(a)=\frac{\log a}{\log b}\cdot\frac{\log b}{\log c}\cdot\frac{\log a}{\log c}=\left(\frac{\log a}{\log c}\right)^2=25$ therefore

$\left(\frac{\log a}{\log c}\right)=\pm5$ From the second equation, $\frac{a^2}{c^2}=c^k\to a^2=c^{k+2}\to a=c^{\frac{k+2}{2}}$

$\left(\frac{\log a}{\log c}\right)=\frac{k+2}{2}=\pm5\therefore k=8,-12$

F06A20 **2.** Without loss of generality, let the length of a side of the new polygon = 1 then the length of a side of the square is $\left(2+\sqrt{2}\right)$. $A_{square}=\left(2+\sqrt{2}\right)^2=6+4\sqrt{2}$

$A_{16-sided}=A_{square}+4A_{triangles}=6+4\sqrt{2}+4\cdot\frac{1}{2}=8+4\sqrt{2}$

$\frac{A_{16-sided}}{A_{square}}=\frac{8+4\sqrt{2}}{6+4\sqrt{2}}\cdot\frac{6-4\sqrt{2}}{6-4\sqrt{2}}=\frac{16-8\sqrt{2}}{4}=4-2\sqrt{2}\to a+b=4+-2=2.$

F06A21 **187.** $\frac{10n+2}{9}=n+21\to n=187.$

F06A22 $\mathbf{\frac{1}{16}}$. Because $\sin(90^\circ-x)=\cos x,\cos(180^\circ-x)=-\cos x$, and $\sin(2A)=2\sin A\cos A$

$\left(\sin165^\circ-\sin75^\circ\right)^2=\left(\sin15^\circ-\cos15^\circ\right)^2$ so $\sin^2 15^\circ+\cos^2 15^\circ-2\sin15^\circ\cos15^\circ=1-\sin30^\circ=1-\frac{1}{2}=\frac{1}{2}.$

Now the given expression equals $\left(\frac{1}{2}\right)^4=\frac{1}{16}.$

F06A23 **0.** $x^2-y^2=2006$ or $(x+y)(x-y)=2006$. The factor pairs of 2006 are (1,2006), (2,1003), (17,108), and (34,59). Let the larger number be $a=x+y$, and the smaller number be $b=x-y\to a+b=2x$. Thus we must find a factor pair with an even sum, and there are no integral solutions.

F06A24 **158.** Drop altitude $\overline{BH}$, let $BH=x$ and $CH=y$ then $AH=21-y$. Using the Pythagorean theorem twice, $x^2+y^2=169$ and $x^2+(21-y)^2=400$. Subtract the two equations and get $441-42y=231\therefore y=5$ and $x=12$.

$\cos A=\frac{16}{20}=\frac{4}{5},\cos C=\frac{5}{13},$

$\cos B=\cos(\sphericalangle ABH+\sphericalangle CBH)=\cos(\sphericalangle ABH)\cos(\sphericalangle CBH)-\sin(\sphericalangle ABH)\sin(\sphericalangle CBH)=$

$\frac{3}{5}\cdot\frac{12}{13}-\frac{4}{5}\cdot\frac{5}{13}=\frac{16}{65}$ $\cos A+\cos B+\cos C=\frac{93}{65}\to p+q=93+65=\mathbf{158}.$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 5

PART I *FALL 2006* *CONTEST 5* *TIME: 10 MINUTES*

F06A25 If $f(x)=\dfrac{3}{x+1}$ and $f(g(x))=\dfrac{6-3x}{6-x}$, then $g(x)$ may be expressed in the form $\dfrac{a}{x+b}$. Compute ab.

F06A26 Compute the number of ways on the coordinate axes, one can travel from the point (2,2) to the point (5,6) with the following stipulations: travel may only be in a vertical or horizontal direction, turns can only be made at points whose coordinates are both integers (lattice points), and if (c,d) is reached after (a,b), then $c \geq a$ and $d \geq b$.

PART II *FALL 2006* *CONTEST 5* *TIME: 10 MINUTES*

F06A27 The product of three consecutive odd integers is fifteen times the sum of the same integers. Compute all possible sums of the three consecutive odd integers.

F06A28 Given: $\log\left(\dfrac{1}{\sin x}\right)+\log\left(\dfrac{1}{\cos x}\right)=2$ and $\log(\sin x+\cos x)=\log\left(\sqrt{\dfrac{p}{q}}\right)$, where p and q are relatively prime positive integers. Compute $p+q$.

PART III *FALL 2006* *CONTEST 5* *TIME: 10 MINUTES*

F06A29 Compute the sum of the terms of the infinite geometric sequence: $(1+\sqrt{2}),(-1),(-1+\sqrt{2}),(-3+2\sqrt{2}),\ldots$

F06A30 In triangle ABC, $17a^2+b^2+9c^2=2ab+24ac$. Compute $\cos B$.

ANSWERS:

F06A25	**8**
F06A26	**35**
F06A27	**±21**
F06A28	**101**
F06A29	$\dfrac{2+\sqrt{2}}{2}$
F06A30	$\dfrac{2}{3}$

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F06A25 **8.**

$f(g)=\dfrac{3}{\dfrac{a}{x+b}+1}=\dfrac{3x+3b}{x+a+b}=\dfrac{6-3x}{6-x}=\dfrac{3x-6}{x-6}$. $\therefore 3b=-6$ and $a+b=-6 \therefore b=-2,\ a=-4$. Thus $ab=8$.

OR: Let $y=g(x)$. Then $\dfrac{3}{y+1}=f(y)=\dfrac{6-3x}{6-x}$. So $\dfrac{y+1}{3}=\dfrac{6-x}{6-3x}$, so $y+1=\dfrac{6-x}{2-x}$, and

$g(x)=y=\dfrac{-4}{x-2}$, etc.

F06A26 **35.** To travel from (2, 2) to (5, 6), we must eventually travel 3 units to the right and 4 units up. Thus we must travel a total of 7 units. Thus we want combinations of RRRUUUU, which is $\dfrac{7!}{3!\cdot 4!}=35$.

F06A27 **±21.** Let the three consecutive odd integers be $x-2$, x, and $x+2$, then the sum is $3x$. The product becomes $(x-2)(x)(x+2)=15(3x)=45x$. Multiplying out gets us,

$x^3-4x=45x$ or $x^3=49x$.. Since x cannot be 0, we can divide by x. $x^2=49 \rightarrow x=\pm 7$. Therefore the sum is -21 or $+21$.

F06A28 **101.** $\log\left(\dfrac{1}{a}\right)=-\log(a) \therefore \log\left(\dfrac{1}{\sin x}\right)+\log\left(\dfrac{1}{\cos x}\right)=2$

$-\log(\sin x)-\log(\cos x)=2$ or $\log(\sin x)+\log(\cos x)=-2$ then $\log(\sin x\cos x)=-2$

So $\sin x\cos x=\dfrac{1}{100}$. $\log(\sin x+\cos x)=\log\left(\sqrt{\dfrac{p}{q}}\right)$. Double both sides and use the properties of logs to obtain:

$\log(\sin x+\cos x)^2=\log\left(\dfrac{p}{q}\right), \log(\sin x+\cos x)^2=\log(\sin^2 x+\cos^2 x+2\sin x\cos x)=\log\left(1+2\left(\dfrac{1}{100}\right)\right)=\log\left(\dfrac{51}{50}\right)=\log\left(\dfrac{p}{q}\right)$

$p+q=51+50=101$.

F06A29 $\mathbf{\dfrac{2+\sqrt{2}}{2}}$. The series is an infinite geometric series with $r=1-\sqrt{2}$. The sum of the sequence is $\dfrac{a}{1-r}=\dfrac{1+\sqrt{2}}{1-(1-\sqrt{2})}=\dfrac{1+\sqrt{2}}{\sqrt{2}}=\dfrac{2+\sqrt{2}}{2}$.

F06A30 $\mathbf{\dfrac{2}{3}}$. $17a^2+b^2+9c^2=2ab+24ac$. By completing the squares we get

$a^2-2ab+b^2+16a^2-24ac+9c^2=0$ or $(a-b)^2+(4a-3c)^2=0$. The only possible solution is $b=a$ and $c=\dfrac{4}{3}a$. Without loss of generality, let $a=3$. Then $b=3$ and $c=4$. Draw the altitude from C to find that $\cos B=\dfrac{2}{3}$.