NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 1

PART I	***SPRING 2011***	***CONTEST 1***	***TIME: 10 MINUTES***

S11B01 The area of a square is numerically equal to its perimeter. Compute the length of the square's diagonal.

S11B02 What ordered pair (p, q) will make the equation

$$(x^2 + px + q)^2 = 1 + (x+1)(x+2)(x+3)(x+4)$$

true for all real values of x?

PART II	***SPRING 2011***	***CONTEST 1***	***TIME: 10 MINUTES***

S11B03 The perimeter of a rhombus is 28 and the area of the rhombus is 25. Compute the sum of the lengths of the diagonals of the rhombus.

S11B04 Alexy and Antonio each sell the same nonzero number of tickets for the homecoming football game. There are adult tickets and student tickets (and no other kind). Alexy sells 3 times as many student tickets as adult tickets, while Antonio sells 3 times as many adult tickets as student tickets. The amount of money that Antonio collects is 1.5 times the amount of money that Alexy collects. Compute the ratio of the price of a student ticket to the price of an adult ticket.

PART III	***SPRING 2011***	***CONTEST 1***	***TIME: 10 MINUTES***

S11B05 In what six-digit number does multiplying by 4 have the same effect as moving its units' digit, which is a 7, from the right side of the number to the far left side of the number, shifting all other digits one place to the right?

S11B06 Compute the probability of obtaining exactly three heads in six tosses of a fair coin if you know for sure that at least one head will appear in the first three tosses.

ANSWERS TO CONTEST 1

S11B01. $4\sqrt{2}$

S11B02. (5, 5)

S11B03. $2\sqrt{74}$

S11B04. $\frac{3}{7}$ or $3:7$

S11B05. 179487

S11B06. $\frac{19}{56}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 1

Spring 2011 Solutions

S11B01 $\mathbf{4\sqrt{2}}$. Let s be the length of one side of the square. We have $s^2 = 4s \Rightarrow s = 4$. Therefore, using the fact that the diagonal of a square creates two 45-45-90 triangles, the length of the diagonal of the square is $4\sqrt{2}$.

S11B02 **(5, 5)**. Two polynomials are equal for all values if and only if all their coefficients are equal, so we can set the corresponding coefficients in each side equal to each other. On the left hand side we have $(x^2+px+q)^2 = x^4 + 2px^3 + (p^2+2q)x^2 + 2pqx + q^2$ and on the right hand side we have $1+(x+1)(x+2)(x+3)(x+4) = x^4 + 10x^3 + 35x^2 + 50x + 25$. As a result, equating the coefficients of the cubic terms gives us $2p = 10 \Rightarrow p = 5$ and equating the coefficients of the quadratic terms yields $25 + 2q = 35 \Rightarrow q = 5$. Therefore, the ordered pair (p, q) is (5, 5).

S11B03 $\mathbf{2\sqrt{74}}$. Since the perimeter of the rhombus is 28, each side of the rhombus has length 7. Let the diagonals of the rhombus equal $2x$ and $2y$. Since the diagonals of a rhombus are perpendicular bisectors of each other, using the Pythagorean Theorem, we have $x^2 + y^2 = 49$. Also, since the area of the rhombus is 25, we can write $\frac{1}{2}(2x)(2y) = 25 \Rightarrow 2xy = 25$. Adding these two equations yields $x^2 + 2xy + y^2 = 49 + 25$, so that $(x+y)^2 = 74$. Hence, $x + y = \sqrt{74}$ and the sum of the lengths of the diagonals is $2x + 2y = 2\sqrt{74}$.

S11B04 $\mathbf{\frac{3}{7}}$ **or 3 : 7**. Let 4n be the number of tickets that Alexy and Antonio each sell, let x be the cost of a student ticket (in dollars) and let y be the cost of an adult ticket (in dollars). Alexy sells 3n student tickets and n adult tickets, and therefore collects $3nx + ny$ dollars. On the other hand, Antonio sells n student tickets and 3n adult tickets and collects $nx + 3ny$ dollars. Since Antonio collects 1.5 times as much money as Alexy, we have $\frac{nx+3ny}{3nx+ny} = \frac{3}{2}$. Factoring and solving for $\frac{x}{y}$ gives

$$\frac{n(x+3y)}{n(3x+y)} = \frac{3}{2} \Rightarrow 2x + 6y = 9x + 3y \Rightarrow 3y = 7x \Rightarrow \frac{x}{y} = \frac{3}{7}.$$

S11B05 **179487.** Let the six-digit number be $ABCDE7$, then work forward from the fact that $ABCDE7 \times 4 = 7ABCDE$. We have

$$\begin{array}{cccccc} A & B & C & D & E & 7 \\ \times & & & & & 4 \\ \hline 7 & A & B & C & D & E \end{array} \Rightarrow \begin{array}{cccccc} A & B & C & D & 8 & 7 \\ \times & & & & & 4 \\ \hline 7 & A & B & C & D & 8 \end{array}$$

$$\begin{array}{cccccc} A & B & C & 4 & 8 & 7 \\ \times & & & & & 4 \\ \hline 7 & A & B & C & 4 & 8 \end{array} \Rightarrow \begin{array}{cccccc} A & B & 9 & 4 & 8 & 7 \\ \times & & & & & 4 \\ \hline 7 & A & B & 9 & 4 & 8 \end{array} \Rightarrow \begin{array}{cccccc} A & 7 & 9 & 4 & 8 & 7 \\ \times & & & & & 4 \\ \hline 7 & A & 7 & 9 & 4 & 8 \end{array} \Rightarrow \begin{array}{cccccc} 1 & 7 & 9 & 4 & 8 & 7 \\ \times & & & & & 4 \\ \hline 7 & 1 & 7 & 9 & 4 & 8 \end{array}$$

Therefore, the number is 179487. You can also obtain this result by representing the six-digit number as $10x + 7$ and solving the equation $4(10x + 7) = 700000 + x$.

S11B06 $\mathbf{\frac{19}{56}}$. Without the condition of at least one head in the first three tosses, there would be $2^6 = 64$ elements in the sample space. However, there are $2^3 = 8$ tosses where the first three tosses are all tails. So, eliminating these possibilities, the sample space has $64 - 8 = 56$ elements. Without the condition of at least one head in the first three tosses, there are ${}_6C_3 = \frac{6!}{3!(6-3)!} = 20$ tosses consisting of 3 heads and 3 tails. Eliminating the toss of three successive tails followed by three heads gives a total of $20 - 1 = 19$ tosses fitting the condition. Therefore, the probability is $\frac{19}{56}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division   CONTEST NUMBER 2

PART I ***SPRING 2011*** ***CONTEST 2*** ***TIME: 10 MINUTES***

S11B07 The sum of the squares of the lengths of all four of the sides of a rectangle is 882. Compute the length of a diagonal of the rectangle.

S11B08 The sum of the positive divisors of 960 is 3048. Compute the sum of the reciprocals of the positive divisors of 960. (Note that 1 and 960 are both positive divisors of 960.)

PART II ***SPRING 2011*** ***CONTEST 2*** ***TIME: 10 MINUTES***

S11B09 Let *A* and *B* represent base 5 digits, not necessarily distinct, such that $B = 2A$. Compute the sum, in base 5, of all base 5 numbers of the form $4A2B_5$.

S11B10 The cubic equation $x^3 - 3x^2 + 2x + k = 0$ has complex roots *p*, *q* and *r*, and $(p+2)(q+2)(r+2) = 17$. Compute *k*.

PART III ***SPRING 2011*** ***CONTEST 2*** ***TIME: 10 MINUTES***

S11B11 A box contains a mixture of yellow and green marbles. If two more yellow marbles and no green marbles were added to the box, then the box would have three times as many yellow marbles as green marbles. If instead, three green marbles and no yellow marbles were added, the box would contain twice as many yellow marbles as green marbles. Compute the original total number of marbles in the box.

S11B12 Triangle *XYZ* has $XY = 7$, $YZ = 8$ and $XZ = 9$. Let *T* be the trisection point of side $\overline{XZ}$ closer to point *Z*. Compute the length of $\overline{YT}$.

ANSWERS TO CONTEST 2

S11B07. 21

S11B08. $\frac{127}{40} = 3\tfrac{7}{40} = 3.175$

S11B09. 22421 (or 22421_5)

S11B10. 7

S11B11. 30

S11B12. $\sqrt{41}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 2

Spring 2011 Solutions

S11B07 **21**. Let w = the width of the rectangle and l = the length of the rectangle. We have $2w^2+2l^2=882 \Rightarrow w^2+l^2=441$. By the Pythagorean theorem, the length of the diagonal of the rectangle is $\sqrt{w^2+l^2}$. So, by substitution, the length of a diagonal is $\sqrt{441}=21$.

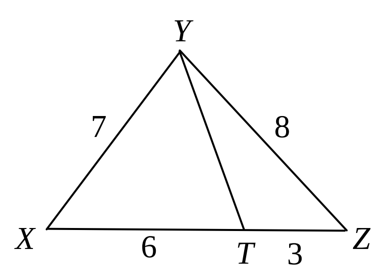
S11B08 $\mathbf{\frac{127}{40}=3\frac{7}{40}=3.175}$. Without finding its prime factorization, some of the divisors of 960 are 1, 2, 3, 4… 240, 320, 480, and 960. The sum of their reciprocals can be written as
$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{240}+\frac{1}{320}+\frac{1}{480}+\frac{1}{960}=\frac{960+480+320+240+...+4+3+2+1}{960}=\frac{3048}{960}=\frac{127}{40}.$$

S11B09 **22421 (or $\mathbf{22421_5}$)**. There are only three possible base-5 numbers $4\underline{A}2\underline{B}_5$ such that $B=2A$: 4020_5, 4122_5 and 4224_5. Keeping track of "carrying," in base-5 the sum of these three numbers is
$$\begin{array}{rrrrr} & 4 & 0 & 2 & 0_5 \\ & 4 & 1 & 2 & 2_5 \\ + & 4 & 2 & 2 & 4_5 \\ \hline 2 & 2 & 4 & 2 & 1_5 \end{array}.$$

S11B10 **7**. From the equation $x^3-3x^2+2x+k=0$, we know that $p+q+r=3$, $pq+pr+qr=2$, and $pqr=-k$. Expanding the left hand side of $(p+2)(q+2)(r+2)=17$, we have $pqr+2(pq+pr+qr)+4(p+q+r)+8=17$. Now, we can find the value of k by using substitution: $-k+2\cdot2+4\cdot3+8=17 \Rightarrow k=7$. Alternately, we know that $(x-p)(x-q)(x-r)=x^3-3x^2+2x+k$ holds for all real values of x, so we can substitute in x = -2. This gives us $-(p+2)(q+2)(r+2)=-8-12-4+k=k-24 \Rightarrow -17=k-24 \Rightarrow k=7$.

S11B11 **30**. Let Y = the number of yellow marbles and G = the number of green marbles. From the given information we have $Y+2=3G \Rightarrow Y=3G-2$ and $G+3=\frac{1}{2}Y$. Using substitution and solving for G, we have $G+3=\frac{1}{2}\left(3G-2\right) \Rightarrow 2G+6=3G-2 \Rightarrow G=8$. As a result, $Y=3\cdot8-2=22$ and the total number of marbles in the box is $8+22=30$.

S11B12 $\mathbf{\sqrt{41}}$. The Law of Cosines can be expressed two ways: $\cos\angle C=\frac{a^2+b^2-c^2}{2ab}$ or $c^2=a^2+b^2-2ab\cos\angle C$, where a, b and c are the lengths of the sides of the triangle and $\angle C$ is the angle opposite side c. Using the first equation with $\angle X$ we have: $\cos\angle X=\frac{7^2+9^2-8^2}{2\cdot7\cdot9}=\frac{66}{126}=\frac{11}{21}$.



Next, using the second case to find YT we have: $YT^2 = 7^2 + 6^2 - 2 \cdot 7 \cdot 6 \cdot \frac{11}{21} = 85 - 44 = 41$. Therefore, $YT = \sqrt{41}$. Alternately, this problem can be solved using Stewart's Theorem: $(6)(9)(3) + 9 \cdot YT^2 = (8)(6)(8) + (7)(3)(7) \Rightarrow 9 \cdot YT^2 = 369 \Rightarrow YT^2 = 41$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 3

PART I ***SPRING 2011*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S11B13 A rectangle has sides of lengths 5 cm and 12 cm. A curve is drawn consisting of all points outside the rectangle that are exactly 2 cm from the nearest point on the rectangle. Compute the area of the region enclosed by this curve (in cm^2).

S11B14 How many four-digit positive integers contain the digit pattern '75' once and only once?

PART II ***SPRING 2011*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S11B15 Zack drove 16 miles at 48 miles per hour, then he drove 20 miles at 40 miles per hour, and he finally drove 24 miles at 36 miles per hour. What was Zack's average speed, in miles per hour, for the entire trip?

S11B16 In triangle *ABC*, $AB = 25$, $BC = 15$ and $AC = 20$. If point *D* is a point on $\overline{AB}$ such that $AD = 10$, then compute the length of $\overline{CD}$.

PART III ***SPRING 2011*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S11B17 The price of a stock increased 25% after one year, and then increased $33\frac{1}{3}\%$ over the new price at the end of the next year. After a third year, the stock was still worth 10% more than its original price. Compute the percent decrease of the stock price after the third year from its price at the end of the second year.

S11B18 In the rectangular coordinate plane, the line $y = \dfrac{2}{3}x$ bisects the acute angle formed by the line $y = kx$ and the positive *x*-axis. Compute *k*.

ANSWERS TO CONTEST 3

S11B13. $128 + 4\pi$

S11B14. 278

S11B15. 40

S11B16. $6\sqrt{5}$

S11B17. 34%

S11B18. $\frac{12}{5} = 2\,\frac{2}{5} = 2.4$

New York City Interscholastic Mathematics League

Senior B Division CONTEST NUMBER 3

Spring 2011 Solutions

S11B13 **$128+4\pi$**. The boundary of the region consists of four quarter-circles and four line segments. The inner region can be divided into 4 quarter-circles and 5 rectangles: the central 5 cm by 12 cm rectangle, two 2 cm by 5 cm rectangles, and two 2 cm by 12 cm rectangles. The area of each quarter-circle is $\frac{\pi}{4}\cdot 2^2=\pi$ so that the area of all four quarter-circles is 4π. The area of the rectangles are 60, $2\times 10=20$, and $2\times 24=48$. Therefore, the area of the region is $60+20+48+4\pi=128+4\pi$ square centimeters.

S11B14 **278.** If '75' occupies the first two positions, the numbers are <u>75</u>00 through <u>75</u>99, which is 100 numbers; excluding 7575 leaves 99 numbers. If '75' occupies the second and third positions, the numbers are 1<u>75</u>0 through 9<u>75</u>9, which, by the Fundamental Principle of Counting, is $9\cdot 10=90$ numbers (nine choices for the first digit, and 10 choices for the last digit). If '75' occupies the last two positions, the numbers are 10<u>75</u> through 99<u>75</u>, which is 90 numbers. However, in this case, excluding 7575 leaves 89 numbers. Therefore, all together there are $99+90+89=278$ numbers.

S11B15 **40**. Average speed is total distance divided by total time. Zack traveled a total of $16+20+24=60$ miles. Zack's total travel time was $\frac{16}{48}=\frac{1}{3}$ hour plus $\frac{20}{40}=\frac{1}{2}$ hour plus $\frac{24}{36}=\frac{2}{3}$ hour for a total of $\frac{3}{2}$ hours. Therefore, Zack's average speed was $\frac{60}{\frac{3}{2}}=40$ miles per hour.

S11B16 **$6\sqrt{5}$**. Triangle *ABC* is a 15-20-25 right triangle. Draw $\overline{DE}$ perpendicular to $\overline{AC}$ so that $\Delta ADE \sim \Delta ABC$ by the AA Similarity Theorem. Hence, ΔADE must be a 6-8-10 right triangle with $DE=6$ and $AE=8$. Since $AE=8$ and $AC=20$, we have $CE=12$. Now, using the Pythagorean Theorem on right triangle CDE, we have $6^2+12^2=CD^2 \Rightarrow CD^2=180 \Rightarrow CD=\sqrt{180}=6\sqrt{5}$. Alternately, we know from right triangle *ABC* that $\cos A=\frac{4}{5}$, so by Law of Cosines $CD^2=10^2+20^2-2(10)(20)\frac{4}{5} \Rightarrow CD^2=500-400\frac{4}{5} \Rightarrow CD^2=180 \Rightarrow CD=6\sqrt{5}$. A third solution uses Stewart's Theorem: $(AD)(AB)(BD)+(CD)^2(AB)=(BC)^2(AD)+(AC)^2(BD)$ $\Rightarrow (10)(25)(15)+(CD)^2(25)=(15)^2(10)+(20)^2(15) \Rightarrow (CD)^2=180 \Rightarrow CD=6\sqrt{5}$. And finally, one more: Note that *BCD* is an isosceles triangle. Let *M* be the midpoint of *CD*, which is also the foot of the altitude from *B*. We know from right triangle *ABC* that $\cos B=\frac{3}{5}$, so using the double angle formulas, we have $\cos B=\frac{3}{5}=1-2\sin^2\frac{B}{2} \Rightarrow \sin\frac{B}{2}=\frac{\sqrt{5}}{5}$. In right triangle *CBM*, $\sin\frac{B}{2}=\frac{\sqrt{5}}{5}=\frac{CM}{15}=\frac{CD}{30}$ $\Rightarrow CD=6\sqrt{5}$.

S11B17 **34%**. In a first year, a 25% increase in the price corresponds to multiplying the price by $\frac{5}{4}$. In the second year, a $33\frac{1}{3}\%$ increase corresponds to multiplying the price by $\frac{4}{3}$. If the price at the end of three years is 10% more than the original price, then the stock is worth $\frac{11}{10}$ of its original price.

If we let *x* equal the ratio of change from the second year to the third year, we have $\frac{5}{4}\cdot\frac{4}{3}\cdot x=\frac{11}{10}\Rightarrow x=\frac{33}{50}=\frac{66}{100}$, and as a result the price of the stock decreased 34% from year two to year three.

S11B18 $\mathbf{\frac{12}{5}=2\frac{2}{5}=2.4}$. Let θ be the angle formed by the line $y=\frac{2}{3}x$ and the positive *x*-axis. Using the slope of the line $y=\frac{2}{3}x$, we have $\tan\theta=\frac{2}{3}$, so that the slope of the line $y=kx$ is $k=\tan 2\theta$.

The double angle formula for tangent $\left(\tan 2\theta=\frac{2\cdot\tan\theta}{1-\tan^2\theta}\right)$ yields $k=\tan 2\theta=\frac{2\cdot\frac{2}{3}}{1-\left(\frac{2}{3}\right)^2}=\frac{\frac{4}{3}}{1-\frac{4}{9}}=\frac{12}{5}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 4

PART I ***SPRING 2011*** ***CONTEST 4*** ***TIME: 10 MINUTES***

S11B19 A mixture is 25% blue paint, 30% yellow paint and 45% water. If 4 quarts of blue paint are added to 20 quarts of the mixture, what is the percentage of blue paint in the new mixture?

S11B20 A regular hexagon has a perimeter of $12\sqrt{3}$ inches. An equilateral triangle is constructed that has a side length equal to the longest diagonal of the hexagon. Compute the area of this triangle in square inches.

PART II ***SPRING 2011*** ***CONTEST 4*** ***TIME: 10 MINUTES***

S11B21 If the area of a circle's inscribed square is 30, what is the area of the circle's circumscribed square?

S11B22 If x and y are positive real numbers such that $x(x-y)=9$ and $y(x-y)=4$, compute $x-y$.

PART III ***SPRING 2011*** ***CONTEST 4*** ***TIME: 10 MINUTES***

S11B23 Compute the value of the positive integer n such that that there are 2011 integers strictly greater than n^2 and strictly less than $(n+2)^2$.

S11B24 One of the diagonals of an 8 by 15 rectangle is drawn, forming two triangles. A circle is inscribed in each triangle. Compute the distance between the centers of the two circles.

ANSWERS TO CONTEST 4

S11B19. $37\frac{1}{2}\% = \frac{75}{2}\% = 37.5\%$

S11B20. $12\sqrt{3}$

S11B21. 60

S11B22. $\sqrt{5}$

S11B23. 502

S11B24. $\sqrt{85}$

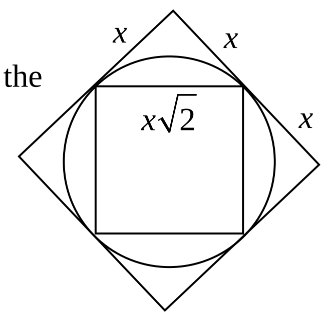
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 4

Spring 2011 Solutions

S11B19 **$37\frac{1}{2}\% = \frac{75}{2}\% = 37.5\%$**. The original mixture is 25% blue paint which means that 5 quarts of the mixture is blue paint since $20 \times 0.25 = 5$. Adding 4 quarts of blue paint to the mixture brings the total amount of blue paint in the new mixture to 9 quarts. Since the new mixture contains 20 + 4 = 24 quarts total, the percentage of blue paint in this mixture is $\frac{9}{24} = \frac{3}{8} = 37.5\%$.

S11B20 **$12\sqrt{3}$**. The longest diagonal of a regular hexagon is twice the length of one of its sides, and the area of an equilateral triangle of side length s is $\frac{s^2}{4}\sqrt{3}$. Since the perimeter of the hexagon is $12\sqrt{3}$, the side of the hexagon is $2\sqrt{3}$. As a result, the longest diagonal measures $4\sqrt{3}$. Therefore, the area of the triangle is $\frac{\left(4\sqrt{3}\right)^2}{4}\sqrt{3} = \frac{48}{4}\sqrt{3} = 12\sqrt{3}$.

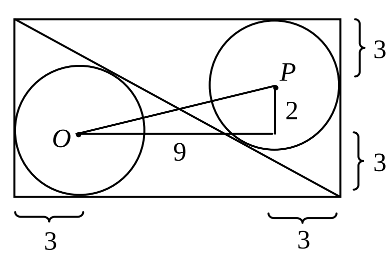


S11B21 **60**. Let $2x$ = the sides of the circumscribed square so that the sides of the inscribed square are $x\sqrt{2}$. Now, $\frac{\text{area of circumscribed square}}{\text{area of inscribed square}} = \left(\frac{2x}{x\sqrt{2}}\right)^2 = \frac{4x^2}{2x^2} = 2$, so that the area of the circumscribed square is $2 \cdot 30 = 60$.

S11B22 **$\sqrt{5}$**. Subtract the two equations to get $x(x-y) - y(x-y) = 5$, then factor and solve for $(x-y)$: $(x-y)(x-y) = 5 \Rightarrow x - y = \sqrt{5}$. Note that $x - y \neq -\sqrt{5}$ since it would make x negative from the first given equation. Alternatively, divide the two equations to get $\frac{x}{y} = \frac{9}{4}$ or $y = \frac{4}{9}x$. Substituting this into the second equation gives $\frac{4}{9}x\left(x - \frac{4}{9}x\right) = 4$. Next, solving this equation for x yields

$\frac{1}{9}x\left(x - \frac{4}{9}x\right) = 1 \Rightarrow \frac{1}{9}x\left(\frac{5}{9}x\right) = 1 \Rightarrow \frac{5}{81}x^2 = 1 \Rightarrow x = \frac{9}{\sqrt{5}}$. Now, $x - y = \frac{5}{9}x = \frac{5}{9} \cdot \frac{9}{\sqrt{5}} = \sqrt{5}$.

S11B23 **502**. Notice that $(n+2)^2 = n^2 + 4n + 4$ is $4n + 4$ greater than n^2. Since there are 2011 positive integers in between the two numbers it must be true that $4n + 4 = 2012 \Rightarrow 4n = 2008 \Rightarrow n = 502$.



S11B24 **$\sqrt{85}$**. The two triangles that are formed are 8-15-17 triangles. One way to find the radius of their inscribed circles is to use the formula $\text{Area of } \Delta = \frac{1}{2} \times \text{Perimeter of } \Delta \times \text{Radius of Circle}$. In this case we have

$\frac{1}{2} \cdot 8 \cdot 15 = \frac{1}{2} \cdot 40 \cdot r \Rightarrow r = 3$. Now, if the centers of the circles are *O* and *P*, the distance between *O* and *P* is the length of the hypotenuse of a triangle with legs of length $15 - 6 = 9$ and $8 - 6 = 2$ (see the diagram). Hence, using the Pythagorean Theorem, $OP = \sqrt{9^2 + 2^2} = \sqrt{85}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 5

PART I ***SPRING 2011*** ***CONTEST 5*** ***TIME: 10 MINUTES***

S11B25 Let f be the linear function such that $f(2x + 4) = 6x + 13$ for all x. Compute the ordered pair (a, b) such that $f(5 - 3x) + f(5x - 1) = ax + b$ for all x.

S11B26 Let $\log_{10} 70 = m$ and $\log_{10} 20 = p$. Given that

$$\log_{10} 14 = Am + Bp + C$$

where A, B, and C are integers, compute the ordered triple (A, B, C).

PART II ***SPRING 2011*** ***CONTEST 5*** ***TIME: 10 MINUTES***

S11B27 Compute all values of a such that the points $(0, -5)$, $(a, -3)$, and $(3, a)$ are collinear.

S11B28 Square $ABCD$ has side length 4. Equilateral triangles ABE and BCF are constructed on the exterior of $ABCD$. Compute the area of triangle DEF.

PART III ***SPRING 2011*** ***CONTEST 5*** ***TIME: 10 MINUTES***

S11B29 A transformation of the plane which maps the point (x, y) onto the point (x', y') is given by

$$x' = 2x - y$$
$$y' = x + 2y$$

Compute the coordinates of the point (a, b) which is mapped onto the point (10, 0) by this transformation.

S11B30 In acute triangle ABC with $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, compute $\sin C$.

ANSWERS TO CONTEST 5

S11B25. (6, 14)

S11B26. (1, 1, -2)

S11B27. -6 and 1

S11B28. $12+8\sqrt{3}$

S11B29. $(4,-2)$

S11B30. $\frac{56}{65}$

New York City Interscholastic Mathematics League
Senior B Division   Contest Number 5
Spring 2011 Solutions

S11B25 **(6, 14)**. First, $f(2x + 4) = 6x + 13 = 3(2x + 4) + 1$, so that $f(x) = 3x + 1$. Next, $f(5 - 3x) + f(5x - 1) = 3(5 - 3x) + 1 + 3(5x - 1) + 1 = 6x + 14$.

S11B26 **(1, 1, -2)**. First, using the fact that logarithms turn products into sums, $m = \log_{10} 70 = \log_{10}(10\cdot 7) = \log_{10} 10 + \log_{10} 7 = 1 + \log_{10} 7$, so that $\log_{10} 7 = m - 1$. Next, in a similar fashion, $p = \log_{10} 20 = \log_{10}(10\cdot 2) = \log_{10} 10 + \log_{10} 2 = 1 + \log_{10} 2$, so that $\log_{10} 2 = p - 1$. Now, $\log_{10} 14 = \log_{10}(7\cdot 2) = \log_{10} 7 + \log_{10} 2 = m - 1 + p - 1 = m + p - 2$. (Challenge: see if you can prove this is the only triple of *integers* that solves the equation. There are infinitely many triples of real numbers that will work.)

S11B27 **-6 and 1**. Label the three points $X(0,-5)$, $Y(a,-3)$, and $Z(3,a)$. For the three points to be collinear, we need the slope of segment $\overline{XY}$ to be equal to the slope of segment $\overline{YZ}$. Hence, $\frac{-3-(-5)}{a-0} = \frac{a-(-3)}{3-a} \Rightarrow \frac{2}{a} = \frac{a+3}{3-a}$. (The points are non-collinear in the cases $a = 0$ and $a = 3$, so there is no division by 0.) Solving for a yields
$6 - 2a = a^2 + 3a \Rightarrow a^2 + 5a - 6 = 0 \Rightarrow (a+6)(a-1) = 0 \Rightarrow a = -6, 1$.

S11B28 $\mathbf{12 + 8\sqrt{3}}$. Sketching a diagram can help. Angles *DAE*, *EBF*, and *DCF* are $150°$ making triangles *ADE*, *BEF* and *CDF* congruent by Side-Angle-Side. As a result, triangle *DEF* is equilateral. Let s = the side length of triangle *DEF*, then using the Law of Cosines we have

$s^2 = 4^2 + 4^2 - 2\cdot 4\cdot 4\cdot \cos 150° = 32 - 32\left(-\frac{\sqrt{3}}{2}\right) = 32 + 16\sqrt{3}$. Now, using the formula for the area of an equilateral triangle $\left(\frac{s^2}{4}\sqrt{3}\right)$, we have that the area of triangle *DEF* is equal to $\frac{32+16\sqrt{3}}{4}\sqrt{3} = 12 + 8\sqrt{3}$.

Alternate solution: Since *ADE* and *CDF* are isosceles triangles with an angle measuring $150°$, angles *ADE* and *FDC* must each measure $15°$, proving that angle *EDF* measures $60°$. *ADE* and *CDF* are congruent by Side-Angle-Side, so $\overline{DE} \cong \overline{DF}$; therefore angles *DEF* and *DFE* are also congruent, making triangle *DEF* isosceles. Let *G* be the foot of the altitude from *F* to the extension of *CD*. *CFG* is a 30-60-90 triangle with $FG = 2$ and $CG = 2\sqrt{3}$. We can now use the Pythagorean theorem to determine that $DF^2 = 2^2 + (4 + 2\sqrt{3})^2 = 32 + 16\sqrt{3}$. By the formula for the area of an equilateral triangle, the area of *DEF* is $DF^2\frac{\sqrt{3}}{4} = \ 32 + 16\sqrt{3}\ \frac{\sqrt{3}}{4} = 12 + 8\sqrt{3}$.

S11B29 **(4,-2)**. From the definition of the transformation, we have $10 = 2a - b$ and $0 = a + 2b$. Multiply the second equation by 2 to get $0 = 2a + 4b$, and subtract this new equation from the first equation to get $10 = -5b \Rightarrow b = -2$.Now, using substitution into the first equation, we have $10 = 2a - (-2) \Rightarrow a = 4$. Therefore, the point that gets mapped onto (10,0) is (4,–2).

S11B30 $\mathbf{\frac{56}{65}}$. Since $\angle A + \angle B + \angle C = 180°$, angle $A + B$ is supplementary to $\angle C$. Therefore, $\sin(A + B) = \sin C$. On the left-hand side, using the angle addition formula for sine and 3-4-5 and 5-12-

13 right triangles: $\sin(A+B)=\sin A\cos B+\sin B\cos A=\frac{3}{5}\cdot\frac{12}{13}+\frac{5}{13}\cdot\frac{4}{5}=\frac{56}{65}\Rightarrow\sin C=\frac{56}{65}$.

Alternatively, triangle ABC can be constructed so that altitude CD has length 15. Then, triangle ACD is a 15-20-25 (3-4-5 times 5) triangle and triangle BCD is a 15-36-39 (5-12-13 times 3) triangle. As a result, the sides of triangle ABC are 39, 25 and 56 (see the diagram).

Using the Law of Sines, we have $\frac{\sin A}{BC}=\frac{\sin B}{AC}=\frac{\sin C}{AB}$

$$\Rightarrow\frac{\frac{3}{5}}{39}=\frac{\frac{5}{13}}{25}=\frac{\sin C}{56}\Rightarrow\sin C=\frac{56}{65}.$$