New York City Interscholastic Mathematics League

JUNIOR DIVISION | **CONTEST NUMBER ONE** | **SPRING 2007**

PART I: 10 minutes | **NYCIML Contest One** | **Spring 2007**

S07J1. Find the smallest positive integer that is both a cube and a multiple of 18.

S07J2. A harmonic progression is one in which the reciprocals of the terms form an arithmetic progression. Given that 4, a, b, and 10 are in harmonic progression, compute $a + b$.

PART II: 10 minutes | **NYCIML Contest One** | **Spring 2007**

S07J3. Compute: $\left(\sqrt[3]{5}-1\right)\left(\sqrt[3]{25}+\sqrt[3]{5}+1\right)$.

S07J4. A box is in the shape of a cube that has edges of length 3 feet. The box is resting on the surface of a horizontal table. Point P is on an edge of the box and is 1 foot above the bottom of the box. Point Q is on the edge furthest from P and is 1 foot below the top of the box. Compute PQ.

PART III: 10 minutes | **NYCIML Contest One** | **Spring 2007**

S07J5. Compute all values of x such that $3x$ % of $2x$ is x.

S07J6. A row of beads consists of 10 red beads and 5 blue beads. Compute the number of possible arrangements of the beads that have no two blue beads adjacent.

ANSWERS:

S07J1.	**216**
S07J2.	**35 / 3**
S07J3.	**4**
S07J4.	$\sqrt{19}$
S07J5.	$0, \frac{50}{3}$. Both Required.
S07J6.	**462**

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SOLUTIONS

S07J1. Notice that $18 = 2 \cdot 3^2$. The exponents of the prime factors of a cube must be multiples of 3, so the requested number is $2^3 \cdot 3^3 = \mathbf{216}$.

S07J2 The problem's conditions imply that 1/4, 1/a, 1/b, and 1/10 are in arithmetic progression. The common difference is $\dfrac{\frac{1}{10}-\frac{1}{4}}{3} = -\dfrac{1}{20}$. Thus the arithmetic progression is 1/4, 1/5, 3/20, 1/10, and so $a + b = 5 + 20/3 = \mathbf{35/3}$.

S07J3. Let $x = \sqrt[3]{5}$. Then the given expression equals $(x-1)(x^2+x+1) = x^3 - 1 = 5 - 1 = \mathbf{4}$.

S07J4. Points P and Q can be thought of as opposite vertices of a box that is 1 by 3 by 3, so $PQ = \sqrt{1^2+3^2+3^2} = \sqrt{\mathbf{19}}$.

S07J5. The problem's condition is equivalent to $\dfrac{3x}{100} \cdot 2x = x$ or $6x^2 = 100x$. Thus $x =$ $\mathbf{0}$, $\boldsymbol{x} = \mathbf{50/3}$.

S07J6. Place the ten red beads in a row first. Each blue bead must now be placed between two red beads or at the beginning or the end of the row of red beads. Thus there are 11 available spaces for the five blue beads, so there are $\binom{11}{5} = \mathbf{462}$ arrangements that have no two blue beads adjacent.

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JUNIOR DIVISION **CONTEST NUMBER TWO** **SPRING 2007**

PART I: 10 minutes **NYCIML Contest Two** **Spring 2007**

S07J7. Compute: $\sqrt{-2} \cdot \sqrt{-3}$.

S07J8. The lengths of the sides of a triangle are 17, 39, and 44. Compute the length of the shortest altitude of the triangle.

PART II: 10 minutes **NYCIML Contest Two** **Spring 2007**

S07J9. Compute the number of ordered pairs (x, y) of positive integers that satisfy $2x + 3y = 2007$.

S07J10. The lengths of the altitudes of a triangle are 6, 8, and h. The possible values for h are $m < h < n$. Compute the ordered pair (m, n).

PART III: 10 minutes **NYCIML Contest Two** **Spring 2007**

S07J11. A cube has edges of length 6 feet. Three of the faces of the cube are *ABCD*, *CDEF*, and *BCFG*. Compute the area of triangle *BDF*.

S07J12. Compute $\left(1+\frac{2}{\sqrt[3]{9}+\sqrt[3]{3}+1}\right)^3$.

ANSWERS:

S07J7.	$-\sqrt{6}$
S07J8.	**15**
S07J9.	**334**
S07J10.	**(24/7, 24)**
S07J11.	$18\sqrt{3}$
S07J12.	**3**

SOLUTIONS

S07J7. $\sqrt{-2}\bullet\sqrt{-3}=i\sqrt{2}\bullet i\sqrt{3}=\mathbf{-\sqrt{6}}$. (we must be careful to remember that $\sqrt{a}\cdot\sqrt{b}\neq\sqrt{ab}$ if a and b are both negative.)

S07J8. Heron's Formula states that $K=\sqrt{s(s-a)(s-b)(s-c)}$, where K is the area of a triangle, a, b, and c are the lengths of its sides, and $s=\frac{1}{2}(a+b+c)$. Thus: $K=\sqrt{50\bullet 33\bullet 11\bullet 6}=\sqrt{25\bullet 2\bullet 11\bullet 3\bullet 11\bullet 3\bullet 2}=2\bullet 3\bullet 5\bullet 11=330$. Let h be the length of the shortest altitude of the triangle. The shortest altitude is the one drawn to the **longest** side, so $\frac{1}{2}\bullet 44\bullet h=330$, and $h=\mathbf{15}$.

S07J9. The equation has integer solutions if and only if y is odd, so $y = 1$ is the least value of y that yields a solution in positive integers. Because $3y = 2007 - 2x$, it follows that $2007 - 2x$ must be a multiple of 3. The greatest possible value of y is therefore: $2001 / 3 = 667$. Thus y can have any odd value between 1 and 667, inclusive, so there are **334** possible values for y, each of which yields a solution in positive integers to the given equation.

S07J10. Let a, b, and c be the lengths of the sides to which the altitudes of lengths 6, 8 and h, respectively, are drawn, and let K be the area of $\triangle ABC$. Then $2K=6a=8b=ch$. Then $a=2K/6$, $b=2K/8$, and $c=2K/h$. The Triangle Inequality implies that $\frac{2K}{6}-\frac{2K}{8}<\frac{2K}{h}<\frac{2K}{6}+\frac{2K}{8}$. Thus $1/24<1/h<7/24$, so $24/7<h<24$ and the ordered pair is $\left(\mathbf{\frac{24}{7},24}\right)$.

S07J11. Because each of the sides of triangle BDF is a diagonal of a face of the cube, triangle BDF is equilateral with sides of length $6\sqrt{2}$. Use the formula for the area of an equilateral triangle, $K=\frac{s^2\sqrt{3}}{4}$, to find that the area of triangle BDF is $\mathbf{18\sqrt{3}}$.

S07J12. Let $x=\sqrt[3]{3}$. Then the given expression is equal to:

$$\left(1+\frac{2}{x^2+x+1}\right)^3=\left(1+\frac{2(x-1)}{x^3-1}\right)^3=\left(1+\frac{2(x-1)}{2}\right)^3=x^3=\mathbf{3}.$$

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JUNIOR DIVISION | **CONTEST NUMBER THREE** | **SPRING 2007**

PART I: 10 minutes | **NYCIML Contest Three** | **Spring 2007**

S07J13. Given that $x = \log 2$, $y = \log 3$ and $z = \log 7$. Express $\log 315$ in terms of x, y, and z, in simplest form without logarithms. (all logs are base 10.)

S07J14. The sum of n consecutive integers is n, where n is an integer and $2 \le n \le 2007$. Compute the number of possible values of n.

PART II: 10 minutes | **NYCIML Contest Three** | **Spring 2007**

S07J15. Compute the smallest positive integer that leaves a remainder of 1 when divided by 6, 8, and 10.

S07J16. Compute $\sum_{k=1}^{89} \log \tan k^0$.

PART III: 10 minutes | **NYCIML Contest Three** | **Spring 2007**

S07J17. The angle-bisector to the hypotenuse of a right triangle divides the hypotenuse into segments whose lengths are 4 and 5. Compute the area of the triangle.

S07J18. Three of the faces of a tetrahedron are right triangles, and their common vertex is the vertex of each of their three right angles. The lengths of the sides of the other face are 9, 10, and 11. Compute the volume of the tetrahedron.

ANSWERS:

S07J13. $1 - x + 2y + z$
S07J14. **1003**
S07J15. **121**
S07J16. **0**
S07J17. $\frac{810}{41}$
S07J18. $5\sqrt{119}$

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JUNIOR DIVISION CONTEST NUMBER THREE SPRING 2007

SOLUTIONS

S07J13.

$\log 315 = \log 5 \cdot 3^2 \cdot 7 = \log 5 + 2\log 3 + \log 7 = \log \frac{10}{2} + 2y + z = \log 10 - \log 2 + 2y + z = \mathbf{1 - x + 2y + z}.$

S07J14 The arithmetic mean of the consecutive integers is $n/n = 1$. In any set of consecutive integers, the mean equals the median, so the median is 1. In general, the median of a set of consecutive integers is itself an integer if and only if the number of consecutive integers is odd. Thus n can be any odd integer between 3 and 2007, inclusive, so there are **1003** possible values of n.

S07J15. The least common multiple of 6, 8, and 10, which is 120, leaves a remainder of 0 when divided by 6, 8, and 10. Thus 1 more than any multiple of 120 will leave a remainder of 1 when divided by 6, 8, and 10. The number meeting the given conditions is therefore **121.**

S07J16. All angles in the solution are measured in degrees.

$$\begin{aligned}\sum_{k=1}^{89} \log \tan k &= \log \tan 1 + \log \tan 2 + \cdots \log \tan 89 \\ &= \log(\tan 1 \tan 2 \cdots \tan 89) \\ &= \log(\tan 1 \tan 89 \tan 2 \tan 88 \cdots \tan 44 \tan 46 \tan 45) \\ &= \log(\tan 1 \cot 1 \tan 2 \cot 2 \cdots \tan 44 \cot 44) = \log 1 = 0\end{aligned}$$

S07J17. The Angle-Bisector Theorem implies that the ratio of the lengths of the legs is 4 : 5. Represent them by $4x$ and $5x$, and conclude that $(4x)^2 + (5x)^2 = (4+5)^2$. Thus $x^2 = 81/41$. The area of the triangle is $\frac{1}{2} \cdot 4x \cdot 5x = 10x^2 = \mathbf{\frac{810}{41}}$.

S07J18. Let a, b, and c be the lengths of the three edges of the tetrahedron that are the legs of the three right triangular faces. Each of the sides of the fourth face is the hypotenuse of one of the right triangles. Thus $a^2 + b^2 = 9^2$, $b^2 + c^2 = 10^2$, and $a^2 + c^2 = 11^2$. Add the three equations and divide by 2 to obtain $a^2 + b^2 + c^2 = 151$, then subtract each of the three original equations from this one to find that $c^2 = 70$, $a^2 = 51$, and $b^2 = 30$. The edge of length a is an altitude of the tetrahedron, and the area of the face to which it is drawn is $\frac{1}{2}bc$, so the volume of the tetrahedron is:

$$\frac{1}{6}abc = \frac{1}{6}\sqrt{51} \cdot \sqrt{30} \cdot \sqrt{70} = \frac{1}{6}\sqrt{3 \cdot 17 \cdot 3 \cdot 10 \cdot 7 \cdot 10} = \frac{1}{6} \cdot 3 \cdot 10\sqrt{7 \cdot 17} = \mathbf{5\sqrt{119}}.$$