New York City
Interscholastic
Mathematics
League

JUNIOR DIVISION | **CONTEST NUMBER ONE** | **FALL 2007**
PART I: 10 minutes | **NYCIML Contest One** | **Fall 2007**

F07J1. If $365(1492x-1732)+31415=31780$, compute the value of $746x-866$.

F07J2. Compute the number of ordered triples (x, y, z) of integers with $1\leq x\leq 100$ and $1\leq y\leq 100$ such that $2x+3y=5z$.

PART II: 10 minutes | **NYCIML Contest One** | **Fall 2007**

F07J3. In ΔABC, $m\angle A=75°$, $m\angle B=60°$, and $AB=12$. Compute BC.

F07J4. Point B is on $\overline{AC}$. If the coordinates of A, B, and C are $(-17, 90)$, (x, y), and $(43, 2)$, respectively, and $AB:BC=2:3$, compute the ordered pair (x, y).

PART III: 10 minutes | **NYCIML Contest One** | **Fall 2007**

F07J5. Compute the number of three-digit integers that have the property that each digit except the last is less than the digit to its right.

F07J6. The area of a right triangle is 20 and its perimeter is 40. Compute the length of the shorter leg.

ANSWERS:

F07J1.	**1/2**
F07J2.	**2000**
F07J3.	$6+6\sqrt{3}$
F07J4.	**(7, 54.8)**
F07J5.	**84**
F07J6.	$\frac{21-\sqrt{281}}{2}$

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JUNIOR DIVISION **CONTEST NUMBER ONE SOLUTIONS** **FALL 2007**

F07J1. Let $y = 746x - 866$. Then $365(2y) + 31415 = 31780$, so $y = ½$.

F07J2. The given equation is equivalent to $2x + 5y = 5z + 2y$, that is, $2(x - y) = 5(z - y)$. Thus the equation has integer solutions if and only if 5 is a divisor of $x - y$. For each integer value of x, there are 20 values of y in the given range so that 5 is a divisor of $x - y$. Thus there are $100 \cdot 20 = 2000$ of the requested triples.

F07J3. Note that $m\angle C = 45°$. Draw altitude $\overline{AH}$. Then ΔABH is 30°-60°-90°, so BH = 6 and $AH = 6\sqrt{3}$. Also, ΔACH is 45°-45°-90°, so $CH = 6\sqrt{3}$, so $BC = 6 + 6\sqrt{3}$.

F07J4. Each of the coordinates of B is the weighted average of the corresponding coordinates of A and C. Thus $x = \dfrac{3 \cdot (-17) + 2 \cdot 43}{5} = 7$, and $y = \dfrac{3 \cdot 90 + 2 \cdot 2}{5} = 54.8$.

F07J5. These numbers must contain three different non-zero digits. Any set of three different non-zero digits can be arranged in precisely one way that fits the conditions, so there are $\binom{9}{3} = 84$ of the desired numbers.

F07J6. Let a and b represent the lengths of the legs, with $a < b$, and let c represent the length of the hypotenuse of the triangle. Then $a + b + c = 40$ implies that $a + b = c - 40$, so $(a+b)^2 = (40-c)^2$. Expand to obtain $a^2 + 2ab + b^2 = 1600 - 80c + c^2$, and then conclude that $2ab = 1600 - 80c$. But ½ ab = 20, so $80 = 1600 - 80c$, and so $c = 19$. Now $a + b = 21$ and $ab = 40$, so a and b are roots of $x^2 - 21x + 40 = 0$. Thus $\{a,b\} = \left\{\dfrac{21 \pm \sqrt{281}}{2}\right\}$, so $a = \dfrac{21 - \sqrt{281}}{2}$.

JUNIOR DIVISION | CONTEST NUMBER TWO | FALL 2007

PART I: 10 minutes | **NYCIML Contest Two** | **Fall 2007**

F07J7. The cost of an item is reduced by 20%. This new cost is then increased by 20% to make the final cost \$60. Compute the number of dollars in the original cost of the item.

F07J8. In ΔABC, D is on $\overline{BC}$ so that $BD:DC = 1:2$, E is on $\overline{AC}$ so that $CE:EA = 1:2$, , and $\overline{AD}$ and $\overline{BE}$ meet at P. Compute $AP:PD$.

PART II: 10 minutes | **NYCIML Contest Two** | **Fall 2007**

F07J9. Compute the values of m so that the following equation has two roots:

$$(m-1)x^2 - 4x + (m+2) = 0$$

F07J10. A sequence of ordered pairs is subject to the following conditions:

- the first ordered pair in the sequence is (0, 0); and
- the last ordered pair in the sequence is (3, 3); and
- if (x, y) is an ordered pair other than the last in the sequence, the next ordered pair must be $(x + 1, y)$, $(x, y + 1)$, or $(x + 1, y + 1)$.

Compute the number of such sequences.

PART III: 10 minutes | **NYCIML Contest Two** | **Fall 2007**

F07J11. Compute the number of perfect square divisors of 729000000.

F07J12. If $x + y + z = 7$, $xy + yz + zx = 8$, and $xyz = 2$, compute the maximum value of x.

ANSWERS:

F07J7. **62.50**
F07J8. **6:1 or 6**
F07J9. 2, -3
F07J10. **63**
F07J11. **64**
F07J12. $3+\sqrt{7}$

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JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 2007**

SOLUTIONS

F07J7. Let x be the original cost in dollars. Then $x \cdot \frac{4}{5} \cdot \frac{6}{5} = 60$, so

$x = 60 \cdot \frac{25}{24} = \frac{125}{2} = 62.50$.

F07J8. Use Mass Points. Assign a mass of 2 to C. Then the masses at A and B are 1 and 4, respectively. Hence the masses at D and E are 6 and 3, respectively, and so $AP : PD = 6$. For more information google mass point geometry.

OR The wording of the problem implies that the requested ratio is the same for any triangle. Assume $\angle C$ is a right angle, and place a coordinate system so that A (0, 3) and B (3, 0). Then E (0, 1) and D (2, 0). Equations of lines BE and AD are $y = -(1/3)x + 1$ and

$y = -(3/2)x + 3$, respectively. Solve to obtain $x = 12/7$. Then AP/PD equals the ratio of the horizontal changes from A to P and P to D, namely, $\frac{12/7}{2 - 12/7} = 6$. (Note that we could have used a parallelogram instead of a square coordinate system, and the reasoning would still be valid, so the result is true for all triangles.)

F07J9. The discriminant is 0 when $16 - 4(m-1)(m+2) = 0$;

$m^2 + m - 6 = 0$, m = 2, -3

F07J10. Each ordered pair in the sequence corresponds to a point in the coordinate plane. Thus each sequence corresponds to a path along lattice points from (0, 0) to (3, 3), where each point can be reached only from a point directly to its left, directly below it, or diagonally from the left and below. The number of paths that can reach any point is therefore the sum of the number of paths that can be reached from the three points described above. Use this idea to label each point in the square lattice from (0, 0) to (3, 3) with the number of paths that reach the point to find that there are 63 such paths.

F07J11. Factor to obtain $729000000 = 2^6 \cdot 3^6 \cdot 5^6$. Every divisor of 729000000 must be of the form $2^a \cdot 3^b \cdot 5^c$, where a, b, and c are nonnegative integers. For the divisor to be a perfect square, a, b, and c must be even. Thus there are 4 choices for each of a, b, and c, and so there are $4 \cdot 4 \cdot 4 = 64$ perfect square divisors.

F07J12. Notice that x, y, and z are roots of $w^3 - 7w^2 + 8w - 2 = 0$. Factor to obtain $(w-1)(w^2 - 6w + 2) = 0$. Thus $\{x, y, z\} = \{1, 3 \pm \sqrt{7}\}$, and the solutions of the given system are the 6 permutations of $(1, 3 - \sqrt{7}, 3 + \sqrt{7})$. The maximum value of x is therefore $3 + \sqrt{7}$.

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JUNIOR DIVISION | **CONTEST NUMBER THREE** | **FALL 2007**
PART I: 10 minutes | **NYCIML Contest Three** | **Fall 2007**

F07J13. Compute $\sqrt{12345^2 + 2 \cdot 12345 \cdot 67890 + 67890^2}$.

F07J14. In ΔABC , $\overline{AT}$ is an angle-bisector. The circle with $\overline{AT}$ as diameter intersects $\overline{AB}$ at P and $\overline{AC}$ at Q, where $Q \neq A$ and $P \neq A$, $AP = 15$, $PB = 7$, and $CQ = 17$. Compute AQ.

PART II: 10 minutes | **NYCIML Contest Three** | **Fall 2007**

F07J15. The cost of an item is decreased by n%, then this new cost is increased by n% to make a final cost of f dollars. If the original cost of the item had been reduced by 5.29%, the final cost would also have been f dollars. Compute n.

F07J16. If the coordinates of A, B, and C are (–14, 1), (–1, 9), and (6, –1), respectively, compute the coordinates of the centroid of triangle ABC. [The *centroid* of a triangle is the point of intersection of its medians.]

PART III: 10 minutes | **NYCIML Contest Three** | **Fall 2007**

F07J17. A train 100 meters long travels at a speed of 40 kilometers per hour. The train passes through a tunnel. The end of the train emerges from the tunnel exactly 12 minutes after the front of the train has entered it. Compute, in kilometers, the length of the tunnel?

F07J18. In ΔABC , AB = 7, BC = 8, CA = 9, $\overline{AD}$ is an angle-bisector, P is the midpoint of $\overline{AD}$, and $\overrightarrow{BP}$ intersects $\overline{AC}$ at E. Compute AE.

ANSWERS:

F07J13.	**80235**
F07J14.	**15**
F07J15.	**23**
F07J16.	**(–3, 3)**
F07J17.	7.9
F07J18.	**63/23**

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JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 2007**
SOLUTIONS

F07J13. Let x = 12345 and y = 67890. Then the given expression equals $\sqrt{x^2+2xy+y^2}=\sqrt{(x+y)^2}=|x+y|=x+y=80235.$

F07J14. Because angles PAT and QAT are congruent inscribed angles, $\overset{\frown}{PT}\cong\overset{\frown}{TQ}$, and because $\overline{AT}$ is a diameter, $\overset{\frown}{APT}$ and $\overset{\frown}{AQT}$ are semicircles. Thus $\overset{\frown}{AP}\cong\overset{\frown}{AQ}$, so $15=AP=AQ.$

F07J15. Let x be the original cost in dollars. Then $x\left(1-\frac{n}{100}\right)\left(1+\frac{n}{100}\right)=(1-.0529)x$, so $1-\frac{n^2}{10000}=1-.0529$. Then $\frac{n^2}{10000}=.0529$, so $n^2=529$, and $n=23$.

F07J16. Let M be the midpoint of $\overline{AB}$. Then each coordinate of M is the average of the corresponding coordinates of A and B. Use the notation $M=\frac{A+B}{2}$ to express this. Let G be the centroid. Because the medians of a triangle are concurrent at a point 2/3 of the way along each median, from the vertex to the opposite side, $G=\frac{2M+C}{3}=\frac{A+B+C}{3}$. Thus each coordinate of G is the average of the corresponding coordinates of A, B, and C, and so the coordinates of G are (–3, 3).

F07J17. If the tunnel is x meters long, the train has taken 12 minutes to travel x + 100 meters. Since 40 kilometers per hour is 2000/3 meters per minute, we have (12)(2000/3) = x + 100, and x = 7900 meters, or 7.9 kilometers.

F07J18. Use the Angle-Bisector Theorem to conclude that $BD : DC = 7 : 9$. Assign masses to B and C of 9 and 7, respectively. Then the mass at D is 16, so the mass at A is 16, and then the mass at E is 23. Thus $AE : EC = 7 : 16$, and so $AE = (7 / 23)AC = 63 / 23$.