NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER ONE** **FALL 2006**

PART I: 10 minutes **NYCIML Contest One** **Fall 2006**

F06J1. The roots of $ax^2+bx+c=0$ are $\dfrac{\sqrt{6}}{\sqrt{6}+\sqrt{2}}$ and $\dfrac{\sqrt{6}}{\sqrt{6}-\sqrt{2}}$, where a, b, and c are integers whose greatest common divisor is 1, and $a>0$. Compute the ordered triple (a, b, c).

F06J2. Working alone, Danny can do a job in 7 hours and Julie can do the same job working alone in 6 hours. They decide to do the job together, but after 1 hour they have an argument and Julie leaves. Compute the number of hours it takes Danny to finish the job.

PART II: 10 minutes **NYCIML Contest One** **Fall 2006**

F06J3. The sum of the first 7 terms of an arithmetic progression is 49, and the sum of the first 17 terms is 289. Compute the 2006$^{\text{th}}$ term.

F06J4. A cube has edges of length 6 feet. Two of the faces of the cube are $ABCD$ and $CDEF$. Point P is on $CDEF$, and is 1 foot from $\overline{CD}$ and 3 feet from $\overline{DE}$. Compute the length of the shortest path along the surface of the cube that joins A and P.

PART III: 10 minutes **NYCIML Contest One** **Fall 2006**

F06J5. In ΔABC, $AB=6$, $BC=8$, $\overline{AH}$ and $\overline{CK}$ are altitudes, and $CK=4$. Compute BH.

F06J6. Compute the number of terms when the expansion of $(a+b+c+d)^{17}$ is expressed in simplest form.

ANSWERS:

F06J1.	**(2, −6, 3)**
F06J2.	$\frac{29}{6}$ or $4\frac{5}{6}$
F06J3.	**4011**
F06J4.	$\sqrt{58}$
F06J5.	$3\sqrt{3}$
F06J6.	**1140**

New York City Interscholastic Mathematics League

JUNIOR DIVISION **CONTEST NUMBER ONE SOLUTIONS** **FALL 2006**

F06J1. The sum of the roots is

$$\sqrt{6}\left(\frac{1}{\sqrt{6}+\sqrt{2}}+\frac{1}{\sqrt{6}-\sqrt{2}}\right)=\sqrt{6}\cdot\frac{\sqrt{6}-\sqrt{2}+\sqrt{6}+\sqrt{2}}{\left(\sqrt{6}+\sqrt{2}\right)\left(\sqrt{6}-\sqrt{2}\right)}=\frac{\sqrt{6}\cdot 2\sqrt{6}}{4}=3$$, and their product

is $\frac{\sqrt{6}}{\sqrt{6}+\sqrt{2}}\cdot\frac{\sqrt{6}}{\sqrt{6}-\sqrt{2}}=\frac{3}{2}$. Thus $c/a=3/2$ and $-b/a=3=6/2$, so the requested ordered triple is **(2, −6, 3)**.

F06J2. Danny does 1/7 of the job in 1 hour and Julie does 1/6 of the job in 1 hour. Working together for 1 hour, they do 1/6 + 1/7 = 13/42 of the job and 29/42 remain.

$$\frac{\frac{1}{7}}{1\text{ hr.}}=\frac{\frac{29}{42}}{x\text{ hrs.}}\rightarrow\frac{1}{7}x=\frac{29}{42}\rightarrow x=\mathbf{\frac{29}{6}}\text{ hours.}$$

F06J3. Use the formula $S_n=\frac{n}{2}\left[2a+(n-1)d\right]$ to obtain $49=\frac{7}{2}(2a+6d)=7(a+3d)$ and $289=\frac{17}{2}(2a+16d)=17(a+8d)$. Divide to get $7=a+3d$ and $17=a+8d$. Solve the system to find that $a=1$ and $d=2$. The 2006th term is thus $1+2005\cdot 2=\mathbf{4011}$.

F06J4. Rotate square $CDEF$ 90° so that it is coplanar with square $ABCD$. The length of a path from A to P along the cube surface is unchanged. Thus the shortest path from A to P is the length of $\overline{AP}$ after the rotation, namely, $\sqrt{7^2+3^2}=\sqrt{58}$.

F06J5. Let K be the area of ΔABC. Then $\frac{1}{2}\cdot 6\cdot 4=K=\frac{1}{2}\cdot 8\cdot AH$, so $AH=3$. $BH^2=6^2-3^2=27\rightarrow BH=\mathbf{3\sqrt{3}}$.

F06J6. The simplified expansion contains all the terms, and only terms, of the form $a^pb^qc^rd^s$, where p, q, r, and s are nonnegative integers such that $p+q+r+s=17$. Thus the number of terms is the number of ordered 4-tuples of nonnegative integers whose sum is 17. This is the same as the number of divisions of 17 objects into 4 groups where some of the groups may have no objects. Represent the objects as x's and the dividers by d's. The requested number of terms is therefore equal to the number of arrangements of 17 x's and 3 d's, namely $\binom{20}{3}=\mathbf{1140}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 2006**

PART I: 10 minutes **NYCIML Contest Two** **Fall 2006**

F06J7. In the town of Imlburg, during the same period that the number of males increased by 18%, the number of females increased by 11%, and the total population increased by 13%. Compute the ratio of males to females at the beginning of this period.

F06J8. In ΔABC, $AB = AC = 25$, $BC = 14$, and $\overline{BM}$ is a median. Compute the area of ΔABM.

PART II: 10 minutes **NYCIML Contest Two** **Fall 2006**

F06J9. Compute the number of consecutive zeros at the end of $(129!)^2$.

F06J10. Compute all ordered pairs (x, y) of positive integers that satisfy:

$x^2 - y^2 + 4x - 2y = -36$

PART III: 10 minutes **NYCIML Contest Two** **Fall 2006**

F06J11. Compute:

$$\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cdots-\cfrac{1}{2-\cfrac{1}{2}}}}}$$

where 2 appears 100 times in the expression.

F06J12. Compute the volume of a regular tetrahedron whose edges have length 6.

ANSWERS:

F06J7. $\frac{2}{5}$
F06J8. **84**
F06J9. **62**
F06J10. **(14, 16), (2, 6). BOTH REQUIRED.**
F06J11. **100 / 101**
F06J12. **$18\sqrt{2}$**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 2006**
SOLUTIONS

F06J7. Let m and f be the number of males and females, respectively, at the beginning of the period. Then $1.18m + 1.11f = 1.13(m+f)$. Simplify to obtain $5m = 2f$. Thus $m/f = 2/5$.

F06J8. Draw altitude $\overline{AH}$, and use the Pythagorean Theorem in ΔABH to conclude that $AH = \sqrt{25^2 - 7^2} = 24$. Thus the area of ΔABC is $\frac{1}{2} \bullet 14 \bullet 24 = 168$. Because the median of a triangle divides it into two triangles of equal area, the area of ΔABM is $\frac{1}{2} \bullet 168 = 84$.

F06J9. A zero will be found at the end when a factor of 5 is multiplied by any even number. As we have plenty of even numbers, we must simply count factors of 5. There are $\left[\frac{129}{5}\right] = 25$ numbers with at least one 5, $\left[\frac{129}{25}\right] = 5$ numbers with an additional 5 and $\left[\frac{129}{125}\right] = 1$ number with a third factor of 5. We thus have 31 zeros at the end of 129! and 62 zeros at the end of $(129!)^2$. ($[x]$ is the greatest integer $\le x$.)

F06J10. Complete the square to obtain $x^2 + 4x + 4 - y^2 - 2y - 1 = -33$. Then $(x+2)^2 - (y+1)^2 = -33$, so $(y+1)^2 - (x+2)^2 = 33$. Let $a = y + 1$ and $b = x + 2$. Then $33 = a^2 - b^2 = (a-b)(a+b)$. Because $a + b = x + y + 3$, it follows that $a + b$ is positive, and so $a - b$ is too. Note that $a - b$ and $a + b$ are integers, and $a - b < a + b$. Thus the possible values of $(a - b, a + b)$ are (1, 33) and (3, 11). The corresponding values of (a, b) are (17, 16) and (7, 4), which yields $(x, y) = (14, 16)$ or (2, 6).

F06J11. Consider the sequence $\frac{1}{2}, \frac{1}{2-\frac{1}{2}}, \frac{1}{2-\frac{1}{2-\frac{1}{2}}}, \cdots$. The first three terms are $1/2, 2/3, 3/4$. To see that this pattern continues, note that for any term x in the sequence, the subsequent term is given by $\frac{1}{2-x}$, and so if the nth term is $\frac{n}{n+1}$, then the next term is $\frac{1}{2-\frac{n}{n+1}} = \frac{n+1}{n+2}$. Thus the term in which 2 appears 100 times is 100/101.

F06J12. Imagine the tetrahedron is placed on a horizontal surface with equilateral ΔABC at the base and D at the top. The altitude from D meets ΔABC at its centroid G. Because AG is 2/3 of the length of the altitude from A in ΔABC, it follows that $AG = 2\sqrt{3}$. Using the Pythagorean Theorem in ΔADG we get $DG = \sqrt{6^2 - \left(2\sqrt{3}\right)^2} = 2\sqrt{6}$. The area of ΔABC is $9\sqrt{3}$, so the volume of the tetrahedron is $\frac{1}{3} \bullet 9\sqrt{3} \bullet 2\sqrt{6} = 18\sqrt{2}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 2006**

PART I: 10 minutes **NYCIML Contest Three** **Fall 2006**

F06J13. Fanny's Ice Cream Shop has 4 different toppings. A sundae with chocolate ice cream costs \$3.00 plus \$.25 per topping. You can get as many toppings as you want or no toppings at all. If each customer in the store chooses a different chocolate sundae and all possible types of chocolate sundaes are ordered, compute the total amount of money they will pay for their sundaes.

F06J14. Compute: $\sqrt{5+\sqrt{24}}+\sqrt{5-\sqrt{24}}$.

PART II: 10 minutes **NYCIML Contest Three** **Fall 2006**

F06J15. Given that p and q are primes such that $p > q$ and $p + q = 7209$, compute $p - q$.

F06J16. In ΔABC, P, Q, and R are on $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively so that $AP = 2$, $PB = 3$, $BQ = 2$, $QC = 4$, $CR = 3$, and $RA = 4$. Compute the ratio of the area of ΔPQR to the area of ΔABC.

PART III: 10 minutes **NYCIML Contest Three** **Fall 2006**

F06J17. Compute all ordered pairs (x, y) of positive integers that satisfy $xy + 2x + 3y = 92$.

F06J18. Compute all real values of a such that $x^2 + ax + a = -8$ has two distinct positive roots.

ANSWERS:

F06J13. **\$56.00**

F06J14. $\mathbf{2\sqrt{3}}$

F06J15. **7205**

F06J16. **2/7**

F06J17. **(4, 12), (11, 5). BOTH REQUIRED.**

F06J18. $-8 < a < -4$

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JUNIOR DIVISION CONTEST NUMBER THREE FALL 2006

SOLUTIONS

F06J13. There is 1 way to choose no toppings, 4 ways to choose 1 topping, 6 ways to choose 2 toppings, 4 ways to choose 3 toppings, and 1 ways to choose 4 toppings. This will cost \$3.00 + \$13.00 + \$21.00 + \$15.00 + \$4.00 = **\$56.00.**

F06J14. Notice that

$$\left(\sqrt{5+\sqrt{24}}+\sqrt{5-\sqrt{24}}\right)^2=5+\sqrt{24}+2\left(\sqrt{5+\sqrt{24}}\right)\left(\sqrt{5-\sqrt{24}}\right)+5-\sqrt{24}=10+2\sqrt{25-24}=12$$

Thus $\sqrt{5+\sqrt{24}}+\sqrt{5-\sqrt{24}}=\sqrt{12}=2\sqrt{3}$.

F06J15. Because $p+q$ is odd, one of p and q must be even. Because they are prime, and because $p>q$, it follows that $q=2$. Thus $p=7207$, and $p-q=7205$.

F06J16. Notice that $\dfrac{[APR]}{[ABC]}=\dfrac{\frac{1}{2}\cdot 2\cdot 4\sin A}{\frac{1}{2}\cdot 5\cdot 7\sin A}=\dfrac{8}{35}$. Similarly, $\dfrac{[BPQ]}{[ABC]}=\dfrac{1}{5}$ and $\dfrac{[CQR]}{[ABC]}=\dfrac{2}{7}$. Thus $\dfrac{[PQR]}{[ABC]}=1-\left(\dfrac{8}{35}+\dfrac{1}{5}+\dfrac{2}{7}\right)=\dfrac{2}{7}$. ($[ABC]$ denotes the area of ΔABC.)

F06J17. The given equation is equivalent to $xy+2x+3y+6=98$. Factor to obtain $(x+3)(y+2)=98$. Because $x+3$ and $y+2$ are positive integers, $(x+3, y+2)$ can only equal (1, 98), (2, 49), (7, 14), (14, 7), (49, 2), or (98, 1). Only (7, 14), and (14, 7) yield ordered pairs (x, y) in which x and y are both positive, namely (4, 12) and (11, 5).

F06J18. The given equation is equivalent to $x^2+ax+(a+8)=0$. In general, a quadratic equation has positive roots if and only if they are both real and both their sum and product is positive. The roots of the equation are real and distinct when $a^2-4(a+8)>0$. Simplify and factor to get $(a+4)(a-8)>0$. This implies that $a>8$ or $a<-4$. The sum of the roots is positive when $a<0$, and the product of the roots is positive when $a+8>0$, that is, when $a>-8$. The three conditions together imply that $-8<a<-4$.