NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Junior Division CONTEST NUMBER 1

PART I ***SPRING 2011*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S11J1 Compute the number of two-digit primes both of whose digits are prime.

S11J2 Find all real values of x that satisfy $(x-4)^{(x-1)^{x+1}} = 1$.

PART II ***SPRING 2011*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S11J3 Find the smallest positive integer multiple of 35 all of whose digits are the same.

S11J4 Three positive integers x, y, and z satisfy $6x = 8y = 9z$. Let $S = x + y + z$. Compute the largest possible value of S that is less than 2011.

PART III ***SPRING 2011*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S11J5 The radius of a sphere is 10. A circle is drawn on the surface of the sphere so that the distance between the centers of the sphere and the circle is 5. Compute the radius of the circle.

S11J6 Find all rational numbers x that satisfy $\sqrt{x-90} = \sqrt[3]{x+90}$.

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S11J1 **4**. The one-digit primes are 2, 3, 5, and 7. Of the 16 two-digit numbers that consist of these digits, four of them are prime, namely, 23, 37, 53, and 73.

S11J2 **1, 3, 5. (ALL REQUIRED)** For real numbers u and v, $u^v = 1$ only if $u = 1$ or $u = -1$ (for suitable values of v) or $v = 0$ (provided u is not 0). For the given equation to be satisfied, $x - 4 = 1$ or $x - 4 = -1$ or $(x-1)^{x+1} = 0$. Thus $x = 5$, $x = 3$, or $x = 1$. Check the latter two values two see that they do in fact satisfy the given equation.

S11J3 **555555**. Because the requested number is divisible by 35, it is divisible by 5, so its last digit must be 0 or 5. But all the digits are the same, so they must all be 5's. The number must also be divisible by 7, so check 5, 55, 555, ... to see that the smallest such number that is divisible by 7 is 555555.

S11J4 **2001**. Let $N = 6x = 8y = 9z$. Then N is a multiple of 6, 8, and 9, so N is a multiple of 72, that is, there is an integer m such that $N = 72m$. Hence $x = 12m$, y = 9m, and z = 8m, so $x + y + z = 29m$. The largest multiple of 29 that is less than 2011 is 2001.

S11J5 $\mathbf{5\sqrt{3}}$. The center of the sphere, the center of the circle, and a point on the circle are the vertices of a right triangle. Use the Pythagorean Theorem to conclude that the radius of the circle is $\sqrt{10^2 - 5^2} = 5\sqrt{3}$.

S11J6 **126**. Let $y = \sqrt{x-90} = \sqrt[3]{x+90}$. Then $y^2 = x - 90$ and $y^3 = x + 90$, so $y^3 - y^2 = 180$, and then $y^3 - y^2 - 180 = 0$. The Rational Root Theorem states that any rational solution of this system must be a factor of -180; thus any rational roots are necessarily integers. In addition, any root y should satisfy the following equation: $y^3 - y^2 = y^2(y-1) = 180$. Testing divisors of 180 that are squares will lead to $y = 6$ as a solution of this equation. Factor to obtain $(y-6)(y^2 + 5y + 30) = 0$. The equation $y^2 + 5y + 30 = 0$ has no real roots so $y = 6$ is the only rational solution of the equation. Substitute to find that $x = 126$.

ANSWERS TO CONTEST ONE

1. 4
2. 1, 3, 5 (ALL REQUIRED)
3. 555555
4. 2001
5. $5\sqrt{3}$
6. 126

New York City Interscholastic Mathematics League
Junior Division — Contest Number 2

Part I ***Spring 2011*** ***Contest 2*** ***Time: 10 Minutes***

S11J7 Compute the number of three-digit positive integers that neither begin nor end with a 1.

S11J8 The lengths of two sides of a triangle are 10 and 14, and its area is 56. Compute the greater of the two possible values for the length of the third side.

Part II ***Spring 2011*** ***Contest 2*** ***Time: 10 Minutes***

S11J9 In rectangle *PQRS*, *T* is on $\overline{PS}$ so that $PT = 2$ and $TS = 3$. The area of triangle *RST* is 7. Find the area of rectangle *PQRS*.

S11J10 If $\sum_{k=1}^{2011} ki^k = a + bi$, where *a* and *b* are real numbers and *i* is the imaginary unit, compute the ordered pair (a, b).

Part III ***Spring 2011*** ***Contest 2*** ***Time: 10 Minutes***

S11J11 Compute the area of a rectangle with perimeter 30 that is inscribed in a circle of radius 7.

S11J12 A rhinoceros moves on the coordinate plane in the following way. It begins at the origin, and it may move from any lattice point (x, y) to $(x, y + 1)$, $(x + 1, y)$, or $(x + 1, y + 1)$. Compute the number of possible paths along which the rhino can travel from the origin to the point whose coordinates are (3, 3).

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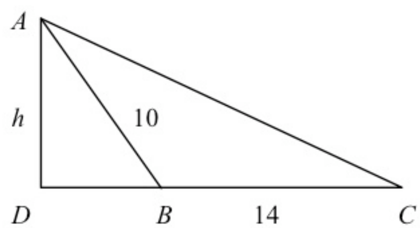
S11J7 **720**. The first digit can be neither 0 nor 1. Thus there are eight choices for the first digit, 10 for the second, and nine for the third for a total of $8{\bullet}10{\bullet}9 = 720$ of the requested integers.

S11J8 $\mathbf{4\sqrt{29}}$. Label the triangle so that AB = 10 and BC = 14, and let h be the length of the altitude from AQ. Then (1/2)(14)h = 56, so h = 8. There are two possible positions for the altitude from A, one inside the triangle and one outside. The latter position will yield the greater length for $\overline{AC}$, the third side. Let D be the foot of the outside altitude from A. Then $BD^2 = AB^2 - h^2 = 10^2 - 8^2$, so BD = 6. Thus $CD = CB + BD$ = 14 + 6 = 20, and so $AC = \sqrt{AD^2 + CD^2} = \sqrt{8^2 + 20^2} = 4\sqrt{2^2 + 5^2} = 4\sqrt{29}$.



Alternate Solution: Let $m\measuredangle ABC$ be t. The area of the triangle is $\frac{1}{2}{\bullet}10{\bullet}14{\bullet}\sin t$, whence sin t = 4/5. Thus cos t = -3/5, since $\measuredangle ABC$ is obtuse. Now apply the Law of Cosines: $AC^2 = 10^2 + 14^2 - 2{\bullet}10{\bullet}14(\cos t) = 296 + 280{\bullet}\frac{3}{5} = 464$. So $AC = 4\sqrt{29}$.

S11J9 **70/3**. In triangle RST, (1/2)(3)(RS)=7, so RS = 14/3. Thus the area of rectangle $PQRS$ is 5(14/3)=70/3.

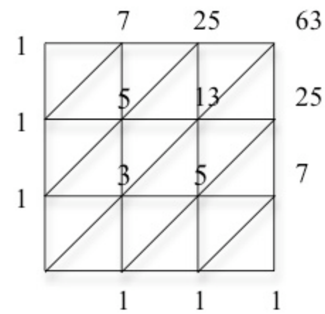
S11J10 **(–1006, –1006)**. Let S equal the given sum. Then

$$S = i + 2i^2 + 3i^3 + 4i^4 + \cdots + 2011i^{2011}$$
$$= \left(\left(-2+4\right)+\left(-6+8\right)+\cdots+\left(-2006+2008\right)-2010\right)+\left(\left(1-3\right)+\left(5-7\right)+\cdots+\left(2009-2011\right)\right)i$$
$$= \left(502{\bullet}2 - 2010\right) + \left(503{\bullet}{-2}\right)i = -1006 - 1006i.$$

Thus (a, b) = (–1006, –1006).

S11J11 **29/2**. Let l and w represent the length and width of the rectangle. Then $2l + 2w = 30$, so $l + w = 15$. Also, because a diagonal of the rectangle is a diameter of the circle, $l^2 + w^2 = 14^2 = 196$. Then $225 = 15^2 = \left(l+w\right)^2 = l^2 + 2lw + w^2$, so $2lw$ = 225 – 196 = 29. Thus the area of the rectangle equals lw = 29/2.

S11J12 **63**. Use the following method to label the grid so that the label of each lattice point is the number of possible paths from the origin to that point. Begin by labeling each lattice point on the axes with a 1. Then use the fact that each point's label is the sum of the labels of the point below it, the point to its left, and the point diagonally to its lower left to label the remaining points, as shown below. Thus there are 63 of the requested paths.


7
25
63
1
5
13
25
1
3
5
7
1
1
1
1

ANSWERS TO CONTEST TWO

1. 720
2. $4\sqrt{29}$
3. 70/3
4. (–1006, –1006)
5. 29/2
6. 63

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division CONTEST NUMBER 3

PART I ***SPRING 2011*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S11J13 Find the greatest prime factor of $2^{20}-1$.

S11J14 The length of a side of a regular hexagon is 1. Compute the sum of the squares of the distances from one vertex to the other five vertices.

PART II ***SPRING 2011*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S11J15 Define a *quaint number* as a composite number that is not divisible by 2, 3, 5, or 7. Compute the sum of the three smallest quaint numbers.

S11J16 Circles C_1 and C_2 intersect at A and B, $A \neq B$. Point P is on C_1, and rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ intersect C_2 at Q and R, respectively. A lies between P and Q, B lies between P and R, and all five points are distinct. If $PA = 4$, $PB = 6$, $BR = 7$, and $AB = 5$, compute RQ.

PART III ***SPRING 2011*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S11J17 The sum of n equal positive real numbers is 1, and the sum of their squares is less than 1/100. Find the minimum value of n.

S11J18 Compute the remainder when $\left\lfloor \left(10+\sqrt{99}\right)^{1000} \right\rfloor$ is divided by 100. (Recall that $\lfloor x \rfloor$ represents the greatest integer that is less than or equal to x.)

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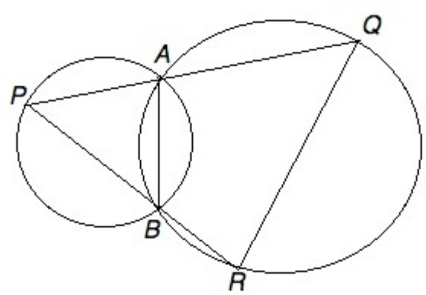
S11J13 **41**. Notice that $2^{20}-1=\left(2^{10}-1\right)\left(2^{10}+1\right)=\left(2^{5}-1\right)\left(2^{5}+1\right)\left(1025\right)=31\bullet 33\bullet 41\bullet 5^2$. Thus the greatest prime factor of $2^{20}-1$ is 41.

S11J14 **12**. Label the hexagon *ABCDEF*. Because each angle of the hexagon measures 120 ° and triangle *BAF* is isosceles, $m\measuredangle BFA$ = 30, and so $m\measuredangle BFE$ = 90. Similarly $m\measuredangle BDE$ = 90. Thus $BE^2=BF^2+FE^2=BF^2+BA^2$, and $BE^2=BD^2+DE^2=BD^2+BC^2$. Add to obtain $2BE^2=BF^2+BA^2+BD^2+BC^2$, and so the requested sum $BF^2+BA^2+BD^2+BC^2+BE^2=3BE^2$. Observe that *BE* = 2, because a hexagon can be partitioned into six equilateral triangles by joining its center to each of the vertices. Thus the requested sum is $3\bullet 2^2=12$.

S11J15 **433**. The prime factors of a quaint number may be 11, 13, 17, 19, ... The candidates for the three smallest are 11•11, 11•13, 11•17, and 13•13. So the three smallest are 121, 143, and 169. Their sum is 433.

S11J16 **65/4**. Triangle *PAB* is similar to triangle *PRQ* because they share $\measuredangle P$ and both $\measuredangle PAB$ and angle *R* are supplementary to $\measuredangle BAQ$. Therefore $\frac{PA}{AB}=\frac{PR}{RQ}$. Thus $\frac{4}{5}=\frac{13}{RQ}$, and so *RQ* = 65/4.



S11J17 **101**. Each of the *n* numbers must equal 1/*n*, and so the sum of their squares is $n\left(1/n^2\right)=1/n$. Thus 1/*n* < 1/100, so *n* > 100, and so the minimum value of *n* is 101.

S11J18 **1**. Notice that

$$\left(10+\sqrt{99}\right)^{1000}+\left(10-\sqrt{99}\right)^{1000}=2\left(10^{1000}+\binom{1000}{2}10^{998}\bullet 99+\binom{1000}{4}10^{996}\bullet 99^2+\cdots+99^{500}\right).$$

Each of the terms within the parentheses is divisible by 100 except for the last term, so the last two digits of the sum are the same as the last two digits of $2\bullet 99^{500}$. But $2\bullet 99^{500}\equiv 2\left(-1\right)^{500}\equiv 2\ \left(\text{mod}\,100\right)$, so the last two digits of the sum are 02. Also, $\left(10-\sqrt{99}\right)^{1000}=\frac{1}{\left(10+\sqrt{99}\right)^{1000}}<\frac{1}{19^{1000}}$. Therefore the remainder when $\left\lfloor\left(10+\sqrt{99}\right)^{1000}\right\rfloor$ is divided by 100 is 1.

ANSWERS TO CONTEST THREE

1. 41
2. 12
3. 433
4. 65/4
5. 101
6. 1