NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division CONTEST NUMBER 1

PART I ***FALL 2010*** ***CONTEST 1*** ***TIME: 10 MINUTES***

F10J1 Compute the smallest number that can be obtained by removing 10 digits from the number 1234512345123451234512345.

F10J2 Each box in a 3-by-3 grid is assigned one the numbers 1, 2, or 3. Compute the number of assignments that have the property that no two numbers in any row are the same and no two numbers in any column are the same.

PART II ***FALL 2010*** ***CONTEST 1*** ***TIME: 10 MINUTES***

F10J3 In triangle ABC, m<A = 64 and m<B = 72. Altitudes $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ intersect at H. Compute the measure of <AHE.

F10J4 Factor $x^4 + x^2 + 25$ as the product of two quadratic polynomials with integer coefficients.

PART III ***FALL 2010*** ***CONTEST 1*** ***TIME: 10 MINUTES***

F10J5 A vertical pole is 1 foot long. The pole is moved one foot from its original position along level ground. It is then moved one foot along the ground from that position in a direction perpendicular to the one in which it originally moved. The pole remains vertical throughout the process. Compute the number of feet in the distance from the top of the pole in its final position to the bottom of the pole in its original position.

F10J6 In trapezoid $ABCD$, $AB = 2$, $BC = 6$, $CD = 10$, and $DA = 6$. Points F and G are the midpoints of $\overline{AD}$ and $\overline{BC}$, respectively, $\overline{FG}$ intersects $\overline{BD}$ and $\overline{AC}$, respectively, at H and J, and $\overline{AC}$ intersects $\overline{BD}$ at E. Compute the area of triangle EHJ.

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Junior Division CONTEST NUMBER 2

PART I ***FALL 2010*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F10J7 Lisa has 22 bags of rocks containing 50 rocks each and 44 bags of rocks containing 10 rocks each. She dumps all the rocks out of the bags and places them in bags containing 3 rocks each until she has made as many bags as she can. Compute the number of rocks that are left over.

F10J8 In trapezoid ABCD with bases $\overline{AB}$ and $\overline{CD}$, $AB = 10$, $BC = 15$, $CD = 18$ and $DA = 9$. Points E and F are on $\overline{AB}$ and $\overline{CD}$, respectively, so that the area of quadrilaterals $AEFD$ and $BEFC$ are equal. If $AE = 4$, compute CF.

PART II ***FALL 2010*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F10J9 Find the closest integer to $\dfrac{2^{1000}+3^{1000}+4^{1000}}{4^{1000}}$.

F10J10 Factor x^4+4y^4 as the product of two quadratic polynomials with integer coefficients.

PART III ***FALL 2010*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F10J11 In a set of three positive integers, the first is one-half the sum of the other two, and the second is one-third the sum of the other two. Compute the least possible sum of the three integers.

F10J12 Compute the number of five-letter combinations that can be made using the letters A, B and / or C. (Changing the order of the letters in a combination does not yield a different combination. For example, the combinations $AABBC$, $ABCBA$, and $CBAAB$ are considered the same.)

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Junior Division CONTEST NUMBER 3

PART I ***FALL 2010*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F10J13 In the game of shmess, the winner gets 4 points and the loser gets 1 point. In the event of a draw, both players get 2 points. Two people play a series of 10 games of shmess, at the end of which they have a total of 46 points. Compute the number of draws there were.

F10J14 In a triangle, the sum of the reciprocals of the lengths of the altitudes is 9/10. Compute the ratio of the area of the triangle to its perimeter.

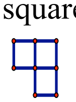
PART II ***FALL 2010*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F10J15 Compute the base of numeration in which $25 \bullet 14 = 317$.

F10J16 Compute the number of sets of four integers, not necessarily distinct, for which the product of the elements is 6.

PART III ***FALL 2010*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F10J17 An L-shaped trionimo consists of three unit squares, one of which shares a side with each of two others, and that are not in a row, as shown in the diagram. Compute the number of positions in which an L-shaped trionimo can be placed on a three-by-three grid, consisting of nine unit squares, so that each square of the trionimo coincides with a square of the grid.



F10J18 Express $7^4 + 4^7$ as a product of prime factors.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division CONTEST NUMBER 1

Fall 2010 Solutions

F10J1 **111231234512345**. After 10 digits are crossed out, the remaining 15-digit number will be smallest when the leftmost digits are smallest. This is achieved by first crossing out the first two occurrences of 2345, and then crossing out the next occurrence of 45. The number that remains is 111231234512345.

F10J2 **12**. Let *abc* represent the order in which the first row is completed, where {a, b, c} = {1, 2, 3}. Then the order of the second row must be *bca* or *cab*. Thus there are six choices for the order of the first row and two for the order of the second row. Notice that there is one choice for the order of the third row after the first two rows have been completed. The number of requested assignments is therefore $6 \bullet 2 \bullet 1 = 12$.

F10J3 **44**. Observe that m<*ACB* = 180 – (64 + 72) = 44. Consider triangle *ACD* to find that m<*CAD* = 180 – (90 + 44) = 46. Then consider triangle *AHE* to conclude that m<*AHE* = 180 – (90 + 46) = 44.
Challenge: Express the measure of each of the six angles with vertex *H* in terms of *A*, *B*, and *C*.

F10J4 $\mathbf{(x^2+3x+5)(x^2-3x+5)}$. Notice that

$$x^4+x^2+25 = x^4+10x^2+25-9x^2 = \left(x^2+5\right)^2-\left(3x\right)^2 = \left(x^2+3x+5\right)\left(x^2-3x+5\right).$$

F10J5 $\mathbf{\sqrt{3}}$. Label the three positions for the bottom of the pole *A*, *B*, and *C*, respectively, and label the final position of the top *D*. Apply the Pythagorean Theorem to triangle *ABC* to conclude that $AC=\sqrt{2}$. Then Apply the Pythagorean Theorem to triangle *ACD* to conclude that $AD=\sqrt{3}$.

F10J6 $\mathbf{4\sqrt{5}/3}$. Let $\overline{AA'}$ be the altitude of trapezoid *ABCD*. Draw $\overline{AP}$ parallel to $\overline{BC}$, with *P* on $\overline{CD}$. Then *ABCP* is a parallelogram, so *AP* = 6, *CP* = 2, and *DA*' = *A*'*P* = 4. Now use the Pythagorean Theorem to conclude that $AA'=2\sqrt{5}$. Since $\triangle ABE \sim \triangle CDE$ and the ratio AB/CD = 1/5, the ratio of the altitude of $\triangle ABE$ to the altitude of $\triangle CDE$ is also 1/5. Thus the altitude of *ABE* is $(1/6)AA'=\sqrt{5}/3$. Given that the base AB = 2, the area of *ABE* is $\left(1/2\right)\left(2\right)\left(\sqrt{5}/3\right)=\sqrt{5}/3$. Because G bisects BC and $\triangle CJG \sim \triangle CAB$, AJ = (1/2) AC. Then EJ = (1/3)AC meaning $EJ = 2 \bullet AE$. Recall that the similarity ratio between the areas of two triangles is the square of the similarity ratio of the corresponding parts of the triangles. So the area of *EHJ* is $(2)^2 \bullet \sqrt{5}/3 = 4\sqrt{5}/3$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division CONTEST NUMBER 2

Fall 2010 Solutions

F10J7 **1**. You can ignore 48 rocks from each 50-rock bag, and you can ignore 9 rocks from each 10-rock bag because they can be divided into bags of three. Also, you can ignore 21 of the 50-rock bags, and you can ignore 42 of the 10-rock bags because they can be divided into bags of three. Therefore the requested number of leftovers is the same as the number of leftovers from one bag of two rocks and two bags of one rock, namely one. Using number congruence notation, these ideas can be expressed as $22 \bullet 50 + 44 \bullet 10 \equiv 1 \bullet 2 + 2 \bullet 1 \pmod 3$.

F10J8 **13/2**. Note that quadrilaterals *AEFD* and *BEFC* are trapezoids with the same height. Call that height *h*. Because the areas of *AEFD* and *BEFC* are equal, conclude that $(1/2)h(AE + DF) = (1/2)h(BE + CF)$. Thus $4 + DF = 6 + CF$, and so $DF - CF = 2$. But $DF + CF = 15$, so $CF = 13/2$.

F10J9 **1**. Note that $\dfrac{2^{1000}+3^{1000}+4^{1000}}{4^{1000}} = \dfrac{2^{1000}}{4^{1000}} + \dfrac{3^{1000}}{4^{1000}} + \dfrac{4^{1000}}{4^{1000}} = \left(\dfrac{2}{4}\right)^{1000} + \left(\dfrac{3}{4}\right)^{1000} + \left(\dfrac{4}{4}\right)^{1000}$.

But $\left(\dfrac{2}{4}\right)^{1000}$ and $\left(\dfrac{3}{4}\right)^{1000}$ are extremely close to 0, and $\left(\dfrac{4}{4}\right)^{1000} = 1$.

F10J10 $\mathbf{(x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)}$. Observe that

$$x^4 + 4y^4 = x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 = (x^2 + 2y^2)^2 - (2xy)^2 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$$

(This is the *Identity of Sophie Germain*.)

F10J11 **12**. Let *a*, *b*, and *c* be the three integers. Then $a = (1/2)(b + c)$ and $b = (1/3)(a + c)$. Hence $2a = b + c$ and $3b = a + c$, so $2a - b = c = 3b - a$, and so $3a = 4b$. Thus $a = 4n$ and $b = 3n$ for some positive integer *n*. Substitute in $2a = b + c$, to find that $c = 5n$. Therefore $a + b + c = 12n$, and the least possible sum is 12.

F10J12 **21**. For any five-letter combination, let *a*, *b*, and *c* be the number of *A*'s, *B*'s and *C*'s, respectively. A pair of combinations is the same if and only if they have the same number of each letter. Thus the requested number of combinations is equal to the number of ordered triples (*a*, *b*, *c*) of nonnegative integers for which $a + b + c = 5$. To determine this number, imagine five tally marks and two dividers arranged in a row. The set of such arrangements are in one-to-one correspondence with the set of ordered triples (*a*, *b*, *c*) of nonnegative integers for which $a + b + c = 5$, and there are $\binom{7}{2} = 21$ of these arrangements.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division CONTEST NUMBER 3

Fall 2010 Solutions

F10J13 **4**. In games in which there is not a draw, a total of five points are scored, and in games in which there is a draw, a total of four points are scored. Let n be the number of games in which there is not a draw and let d be the number of games in which there is a draw. Then $n + d = 10$ and $5n + 4d = 46$. Solve this system to obtain $(n, d) = (6, 4)$.

F10J14 **5/9**. Let K be the area of the triangle, let a, b, and c be the lengths of the sides of the triangle, and let h_a, h_b, and h_c be the lengths of the altitudes to the respective sides. Then $K = (1/2)ah_a = (1/2)bh_b = (1/2)ch_c$, so $1/h_a = a/2K$, $1/h_b = b/2K$, and $1/h_c = c/2K$. Add to obtain $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{2}\left(\frac{a+b+c}{K}\right)$. Thus $9/10 = (1/2)(p / K)$, where p is the perimeter of the triangle, and so $K / p = 5/9$.

F10J15 **13**. Let b be the base of numeration. Then $(2b+5)(b+4)=3b^2+b+7$, so $2b^2+13b+20=3b^2+b+7$, and so $b^2-12b-13=0$. Factor to obtain $(b-13)(b+1)=0$. Thus $b = 13$.

F10J16 **10**. The product of the absolute values of the four integers must be 6, so their absolute values must consist of three 1's and a 6 or two 1's, a two and a three. For the product of the set of four integers to be positive, the set must contain 0, 2 or 4 negative numbers. First consider the case where the absolute values consist of three 1's and a 6. There is one way to assign 0 negatives, two ways to assign two negatives, and one way to assign four negatives for a total of four assignments. Next consider the case where the absolute values consist of two 1's, a two and a three. There is one way to assign 0 negatives, four ways to assign two negatives, and one way to assign four negatives for a total of six assignments. The requested number of sets is therefore 4 + 6 = 10.

F10J17 **16**. For each two-by-two grid, there are four possible allowable positions for an L-shaped trionimo, and the three-by-three grid contains four two-by-two grids. Thus there are 16 positions for the L-shaped trionimo in a three-by-three grid.

F10J18 $\mathbf{5 \bullet 13 \bullet 17^2}$. Notice that $7^4+4^7=7^4+4\bullet 4^6=7^4+4\bullet 2^{12}=7^4+4\bullet 8^4$. Because $x^4+4y^4=(x^2+2xy+2y^2)(x^2-2xy+2y^2)$, conclude that

$$7^4+4^7=7^4+4\bullet 8^4=(7^2+2\bullet 7\bullet 8+2\bullet 8^2)(7^2-2\bullet 7\bullet 8+2\bullet 8^2)=289\bullet 65=5\bullet 13\bullet 17^2.$$

ANSWERS TO CONTEST ONE

1. 111231234512345
2. 12
3. 44
4. $(x^2+3x+5)(x^2-3x+5)$
5. $\sqrt{3}$
6. $4\sqrt{5}\,/\,3$

ANSWERS TO CONTEST TWO

7. 1
8. 13/2
9. 1
10. $(x^2+2xy+2y^2)(x^2-2xy+2y^2)$
11. 12
12. 21

ANSWERS TO CONTEST THREE

13. 4
14. 5/9
15. 13
16. 10
17. 16
18. $5 \cdot 13 \cdot 17^2$