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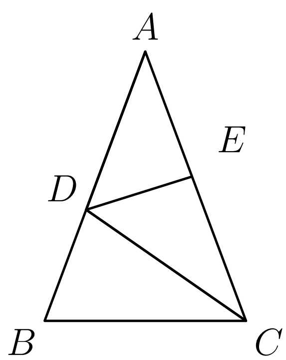
Spring 2013 Sophomore-Freshman Division
Contest Number 1

Problem 1. If x is the smallest prime factor of 2013, compute

$$x \cdot (x+8) \cdot (x+58)$$

Problem 2. In $\triangle ABC$, $\angle BAC = 48^\circ$ and $\angle ABC = \angle ACB$. If $\overline{DC}$ bisects $\angle ACB$ and $\overline{DE}$ bisects $\angle ADC$, compute $\angle DEA$ (in degrees).



Problem 3. The average of nine distinct positive integers is 36, and the median of the set is 26. Compute the maximum possible range of the nine numbers.

Problem 4. If two functions are given as $f(x) = x^2 - 10x + 30$ and $g(x) = 4x^2 - 4x + 6$, compute the sum of all roots for $h(x) = [f(x)]^2 - 2 \cdot [f(x)] \cdot [g(x)] + [g(x)]^2$.

Problem 5. There are four buckets and ten indistinguishable balls. If each bucket must contain at least one ball, compute the number of ways of distributing the ten balls to the four buckets.

Problem 6. Four positive integers a, b, c, and d satisfy the following five conditions:

$$\begin{aligned} a^7 - b^8 &= 0 \\ c^5 - d^4 &= 0 \\ c^2 - 2ac + a^2 &= 225 \\ c &> a \\ \sqrt{a} + \sqrt{c} &< 15 \end{aligned}$$

Compute $a + b + c + d$.

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Spring 2013 Sophomore-Freshman Division
Contest Number 2

Problem 1. Compute the sum of integers that satisfy the following equation:

$$3 \cdot \sqrt{x} - x = \sqrt{x}$$

Problem 2. Compute x that satisfies both $|x - 9| < 6$ and $x^2 = 2x + 15$.

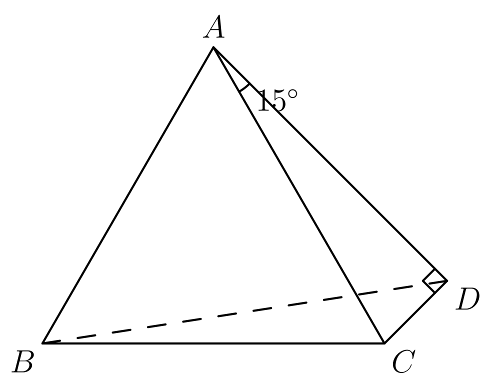
Problem 3. Three distinct numbers are chosen at random from the first 100 positive integers. Compute the probability that their sum is odd.

Problem 4. Real numbers x and y satisfy the following two equations:

$$\begin{aligned} x + xy - y &= 30 \\ -xy^2 + x^2y &= 225 \end{aligned}$$

Compute $x^2 - 2xy + y^2$.

Problem 5. In the following diagram, $\triangle ABC$ is an equilateral triangle with a side length of 8, $\overline{BD} = 10$, and $\angle CAD = 15°$. $\overline{CD}$ can be expressed as $2\sqrt{a} - 4\sqrt{b}$ where both a and b are positive integers. Compute $a + b$.



Problem 6. If $x = 2013^6 + 21^{2012}$, compute the units digit of $x^{2013} + 3^{4x+3}$.

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Spring 2013 Sophomore-Freshman Division
Contest Number 3

Problem 1. a and b are both non-zero integers, and $-100a = 101b$. Compute the smallest possible positive value for $a + b$.

Problem 2. Two functions $f(x) = 2x + 3$ and $g(x) = x^2$ intersect at (a, b) and (c, d). Compute the product, $a \cdot b \cdot c \cdot d$.

Problem 3. When a positive integer x is divided by 21, the remainder is 10. Compute the sum of the remainders when x is divided by 3 and 7.

Problem 4. There are three points on a plane, $A(-4, -1)$, $B(2, 5)$, and $C(a, b)$. The distance between C and the origin is 5, and $\triangle ABC$ cannot be constructed. Compute the sum of all possible coordinates of C.

Problem 5. Let $f_1(x) = 2x + 1$ and $f_n(x) = 2f_{n-1}(x) + 1$ for $n \geq 2$. Compute the sum of the coefficients of $f_9(x)$.
HINT: The sum of a geometric series is:

$$1 + a + a^2 + \cdots + a^b = \frac{a^{b+1} - 1}{a - 1}$$

Problem 6. One of the roots of $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ is $x = \frac{1}{2-\sqrt[4]{5}}$. One of the roots of $g(x) = jx^3 + kx^2 + lx + m$ is $x = \frac{5}{1-\sqrt[3]{3}}$. If $a > 0$ and $j > 0$, compute the sum of the coefficients of $h(x) = f(x) - g(x)$.

Spring 2013 Sophomore-Freshman Division
Contest 1 SOLTUIONS

Problem 1. **2013.** Since 2013 is an odd number, it is not divisible by 2. The sum of all the digits of 2013 $(2+0+1+3=6)$ is divisible by 3 indicating that 2013 is divisible by 3. Therefore, 3 is the smallest prime factor of 2013 and

$$x \cdot (x+8) \cdot (x+58) = 3 \cdot 11 \cdot 61 = 2013$$

Problem 2. **82.5.** Since $\angle BAC = 48°$, $\angle ABC = \angle ACB = \frac{180°-48°}{2} = 66°$. Since $\overline{DC}$ bisects $\angle ACB$, $\angle DCB = 33°$.

$$\begin{aligned} \angle BDC &= 180° - (\angle ABC + \angle DCB) \\ &= 180° - (66° + 33°) \\ &= 81° \end{aligned}$$

Then we can compute $\angle ADC$:

$$\angle ADC = 180° - \angle BDC = 180° - 81° = 99°$$

$$\angle ADE = \frac{\angle ADC}{2} = 49.5°$$

$$\angle DEA = 180° - (\angle BAC + \angle ADE) = 180° - (48° + 49.5°) = 82.5°$$

Problem 3. **203.** Let us denote the nine numbers as $a_1, a_2, a_3, \ldots, a_9$. Without loss of generality, assume that $a_1 < a_2 < a_3 < \cdots < a_8 < a_9$. Then the median of this set is $a_5 = 26$. Since the average is 36, we have

$$a_1 + a_2 + a_3 + a_4 + 26 + a_6 + a_7 + a_8 + a_9 = 324$$

In order to maximize the range, we need to minimize a_1 and maximize a_9:

$$1 + 2 + 3 + 4 + 26 + 27 + 28 + 29 + a_9 = 324 \Longrightarrow a_9 = 204$$

Therefore, the maximum possible range is $a_9 - a_1 = 204 - 1 = 203$.

Problem 4. **-4.** The trick is to complete the square first:

$$\begin{aligned} f(x) &= x^2 - 10x + 30 = (x-5)^2 + 5 \\ g(x) &= 4x^2 - 4x + 6 = (2x-1)^2 + 5 \end{aligned}$$

Notice that $h(x)$ can be factored as

$$h(x) = [f(x)]^2 - 2 \cdot [f(x)] \cdot [g(x)] + [g(x)]^2 = [f(x) - g(x)]^2$$

Substituting $f(x)$ and $g(x)$ into $h(x)$, we get:

$$\begin{aligned} h(x) &= [f(x) - g(x)]^2 \\ &= [(x-5)^2 + 5 - (2x-1)^2 - 5]^2 \\ &= [(x-5)^2 - (2x-1)^2]^2 \\ &= [(3x-6)(-x-4)]^2 \end{aligned}$$

Therefore, $h(x)$ has four roots: 2 and -4, both of multiplicity 2. Thus, the sum is $2+2-4-4 = -4$.

Problem 5. **84.** First, start with one ball in each bucket. Then we need to compute the number of ways of distributing 6 balls to the four buckets. We can use the method of "Stars and Bars" to solve this problem.

There are 6 balls (stars) and 4 buckets, which require $4-1=3$ divisions (bars). In other words, there are $6+3=9$ places for stars and bars, and the problem is reduced to finding the number of ways of choosing 3 places for bars (or alternatively choosing 6 places for stars):

$$\binom{9}{3} = \binom{9}{6} = \frac{9!}{6!3!} = 84$$

Problem 6. $\boxed{\textbf{50.}}$ Rearranging the given equations, we have

$$\begin{aligned} a^7 &= b^8 \\ c^5 &= d^4 \\ (c-a)^2 &= 15^2 \Longrightarrow c-a=15 \\ \sqrt{a}+\sqrt{c} &< 15 \end{aligned}$$

Since a and b are positive integers, we can let $a = x^8$ for a non-zero integer x. Similarly, we can define $c = y^4$ for a non-zero integer y. Substituting these into the third condition, we get:

$$c - a = y^4 - x^8 = 15$$

Factoring the above equation results in:

$$(y^2 + x^4)(y - x^2)(y + x^2) = 15$$

Notice that 15 has four factors: 1, 3, 5, and 15. Since $y - x^2 < y + x^2 < y^2 + x^4$ and $y^2 + x^4 = \sqrt{a} + \sqrt{c} < 15$, we have

$$\begin{aligned} y - x^2 &= 1 \\ y + x^2 &= 3 \\ y^2 + x^4 &= 5 \end{aligned}$$

Solving these equations, we get $x = 1$ and $y = 2$. Then $a = b = 1$, $c = 16$, and $d = 32$.

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Spring 2013 Sophomore-Freshman Division
Contest 2 SOLTUIONS

Problem 1. $\boxed{\textbf{4.}}$ Simplifying the equation, we have

$$x = 2\sqrt{x}$$

Squaring both sides and factoring give us:

$$x^2 = 4x \Longrightarrow x(x-4) = 0$$

The two integers that satisfy the given equation are $x = 0$ and $x = 4$, and the sum of these two values is 4.

Problem 2. $\boxed{\textbf{5.}}$ From the inequality, we have

$$|x - 9| < 6 \Longrightarrow 3 < x < 15$$

Factoring the given equation results in:

$$\begin{aligned} x^2 - 2x - 15 &= 0 \\ (x-5)(x+3) &= 0 \\ x &= -3 \text{ or } 5 \end{aligned}$$

Only $x = 5$ satisfies both conditions.

Problem 3. $\boxed{\textbf{49/198.}}$ There are $100 \cdot 99 \cdot 98$ ways of choosing three distinct positive integers from the first 100 positive integers. In order for the sum to be odd, either all three numbers must be odd or exactly one number out of the three must be odd.

Since there are 50 odd numbers in the first 100 positive integers, there are $50 \cdot 49 \cdot 48$ ways of choosing all three odd numbers. Similarly, there are $50 \cdot 49 \cdot 50$ ways of choosing two even numbers and one odd number. Then the probability for the sum to be odd is:

$$\frac{50 \cdot 49 \cdot 50 + 50 \cdot 49 \cdot 48}{100 \cdot 99 \cdot 98} = \frac{49 \cdot 50 + 49 \cdot 48}{2 \cdot 99 \cdot 98} = \frac{49}{198}$$

Problem 4. $\boxed{\textbf{225.}}$ Let us denote $A = xy$ and $B = x - y$. Then we can rewrite the given equations as

$$\begin{aligned} A + B &= 30 \\ A \cdot B &= 225 \end{aligned}$$

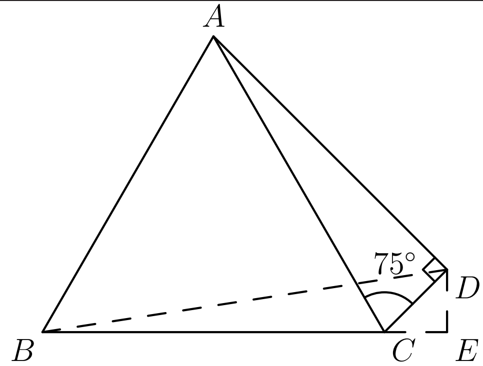
Substituting $A = 30 - B$ into $A \cdot B = 225$ results in

$$(30 - B) \cdot B = 225 \Longrightarrow B^2 - 30B + 225 = 0 \Longrightarrow (B - 15)^2 = 0$$

Therefore, $B = 15$. Since $x^2 - 2xy + y^2 = (x - y)^2 = B^2$, the answer is $B^2 = 15^2 = 225$.

Problem 5. $\boxed{\textbf{19.}}$ Extending $\overline{AD}$ and $\overline{BC}$, we have the following diagram. Since $\triangle ABC$ is an equilateral triangle, we have $\angle ACB = 60°$ and $\angle DCE = 45°$. Then $\triangle CED$ is an isosceles triangle with $\overline{CE} = \overline{DE}$. Let us denote $\overline{CE} = x$. Using the Pythagorean theorem, we have the following equation:

$$\begin{aligned} BE^2 + \overline{ED}^2 &= \overline{BD}^2 \\ (8 + x)^2 + x^2 &= 10^2 \\ x^2 + 8x - 18 &= 0 \end{aligned}$$



Using the quadratic formula, we can solve for x:

$$x = -4 \pm \sqrt{34}$$

But since x has to be positive, $x = -4 + \sqrt{34}$. Then $\overline{CD}$ is:

$$\overline{CD} = x\sqrt{2} = 2\sqrt{17} - 4\sqrt{2}$$

Therefore, $a = 17$ and $b = 2$.

Problem 6. $\boxed{7.}$ The units digit of 2013^6 is equivalent to the units digit of 3^6. Since the units digit of powers of three repeats every four times (3, 9, 7, 1), the units digit of 3^6 is 9. Similarly, the units digit of 21^{2012} is 1. Hence, the units digit of x is same as the units digit of $9 + 1 = 10$, which is 0. From this, we can deduce that the units digit of x^{2013} is 0. Now the problem is reduced to finding the units digit of 3^{4x+3}. Again, the units digit of powers of three repeats every four times, and since the remainder when $4x + 3$ is divided by 4 is 3, the units digit of 3^{4x+3} is 7.

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Spring 2013 Sophomore-Freshman Division
Contest 3 SOLTUIONS

Problem 1. **1.** Since 100 and 101 are coprime numbers (i.e. 1 is the only common divisor between the two numbers), a has to be a multiple of 101 and b has to be a multiple of 100. It is easy to see that $a + b = 1$ when $a = 101$ and $b = -100$.

Problem 2. **-27.** The two functions intersect when

$$\begin{aligned} f(x) &= g(x) \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \Longrightarrow x = -1, 3 \end{aligned}$$

Therefore, the two intersection points are $(-1, 1)$ and $(3, 9)$.

Problem 3. **4.** Let a be the quotient when x is divided by 21. Then we can express x as $x = 21a + 10$. When x is divided by 3, we have

$$\frac{x}{3} = \frac{21a + 10}{3} = 7a + \frac{10}{3}$$

The remainder when x is divided by 3 is then equivalent to the remainder when 10 is divided by 3 which is equal to 1. Similarly, when x is divided by 7, the remainder is 3.

$$\frac{x}{7} = \frac{21a + 10}{7} = 3a + \frac{10}{7}$$

Therefore, the sum of the remainders is 4.

Problem 4. **0.** Since the distance between $C(a, b)$ and $(0, 0)$ is 5, we have the following equation:

$$\sqrt{a^2 + b^2} = 5$$

$\triangle ABC$ cannot be constructed only if C lies on the line that passes through both A and B. From the coordinates of A and B, it is easy to find that both points pass through $y = x + 3$. Since C must be on this line, we have another equation to work with:

$$b = a + 3$$

Substituting this equation into the first equation above and solving for a give us:

$$\begin{aligned} \sqrt{a^2 + (a+3)^2} &= 5 \\ a^2 + 3a - 8 &= 0 \\ a = \frac{-3 \pm \sqrt{41}}{2}, b &= \frac{3 \pm \sqrt{41}}{2} \end{aligned}$$

Therefore, $a + b = 0$.

Problem 5. **1023.** By writing out f_n for small n, we can see that

$$\begin{aligned} f_1(x) &= 2x + 1 \\ f_2(x) &= 2(2x+1) + 1 = 2^2x + 2 + 1 \\ f_3(x) &= 2(2(2x+1)+1) + 1 = 2^3x + 2^2 + 2 + 1 \\ &\vdots \\ f_n(x) &= 2^nx + 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \end{aligned}$$

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The sum of the coefficients of $f_n(x)$ is equivalent to $f_n(1)$:

$$f_n(1) = 2^n + 2^{n-1} + \cdots + 2 + 1$$

Therefore, the sum of the coefficients of $f_9(x)$ is:

$$\begin{aligned} f_9(1) &= 2^9 + 2^8 + 2^7 + \cdots + 2 + 1 \\ &= \frac{2^{10} - 1}{2 - 1} = 2^{10} - 1 = 1023 \end{aligned}$$

Problem 6. $\boxed{\textbf{-71.}}$ Starting with the root of $f(x)$, we have

$$\begin{aligned} x &= \frac{1}{2 - \sqrt[4]{5}} \\ x \cdot \sqrt[4]{5} &= 2x - 1 \\ 5x^4 &= (2x-1)^4 \\ 5x^4 &= 16x^4 - 32x^3 + 24x^2 - 8x + 1 \\ f(x) &= 11x^4 - 32x^3 + 24x^2 - 8x + 1 \end{aligned}$$

Similarly for $g(x)$, we have

$$\begin{aligned} x &= \frac{5}{1 - \sqrt[3]{3}} \\ x\sqrt[3]{3} &= x - 5 \\ 3x^3 &= (x-5)^3 \\ 3x^3 &= x^3 - 15x^2 + 75x - 125 \\ g(x) &= 2x^3 + 15x^2 - 75x + 125 \end{aligned}$$

Therefore, $h(x) = 11x^4 - 34x^3 + 9x^2 + 67x - 124$. The sum of the coefficients is then -71.