SENIOR B DIVISION CONTEST NUMBER ONE SPRING 2013

PART I: 10 MINUTES

S13B1 If $x^2 - y^2 = 77$ and $x + y = 11$, find $x^2 + y^2$.

S13B2 Express in simplest form: $\sum_{n=1}^{\infty} 2 \cdot (0.9)^n$.

PART II: 10 MINUTES NYCIML Contest One Spring 2013

S13B3 In ΔABC, the interior angle bisector of $\measuredangle ABC$ intersects $\overline{AC}$ at point D. If $AB:BC = 3:4$ and $AC = 42$, find AD.

S13B4 Given ΔABC with point R on $\overline{AB}$, point S on $\overline{AC}$ and point T on $\overline{BC}$ such that quadrilateral $BTRS$ is a rhombus. If $AB = 30$ and $BC = 20$, find TS.

PART III: 10 MINUTES NYCIML Contest One Spring 2013

S13B5 When each of the numbers $502, 1021$, and 2405 is divided by the same integer n, where n is greater than one, the remainder r is the same. Find r.

S13B6 In $\Delta ABC, AB = 13, AC = 12$, and $BC = 5$. If $|\cos A + \cos 2B + \sin 3C|$ is expressed in simplest form as ${}^{p}/_{q}$, find $p + q$.

ANSWERS

1. 85
2. 18
3. 18
4. 12
5. 156
6. 301

SENIOR B DIVISION CONTEST NUMBER TWO SPRING 2013

PART I: 10 MINUTES

S13B7 If $f(8x+5)=4x-1$, find $f(29)$.

S13B8 The ratio of the area of an equilateral triangle to the area of a regular hexagon is 24:1. Find the ratio of the length of a side of the equilateral triangle to the length of a side of the regular hexagon.

PART II: 10 MINUTES NYCIML Contest Two Spring 2013

S13B9 Given rectangle $PQRS$ with point T on $\overline{PQ}$ such that $QT=1, PS=9$, and $ST=RS$. Find RS.

S13B10 Evaluate $\sec\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}(1)\right)$.

PART III: 10 MINUTES NYCIML Contest Two Spring 2013

S13B11 If $f(x)=\frac{3}{x-2}$ and $g(x)=4x$, then find all x in simplest form such that $f\big(g(x)\big)=g\big(f(x)\big)$.

S13B12 In an 18 by 24 rectangle, one diagonal is drawn and a circle is inscribed in each of the two right triangles formed. Find the distance between the centers of the circles in simplest radical form.

<u>ANSWERS</u>

1. 11
2. 12
3. 41
4. 2
5. $^2/_5$
6. $6\sqrt{5}$

SENIOR B DIVISION CONTEST NUMBER THREE SPRING 2013

PART I: 10 MINUTES

S13B13 A twelve foot by fifteen foot surface will be covered by tiles. Each of the tiles measure four inches by six inches. How many of these four inch by six inch tiles are necessary to cover the entire surface?

S13B14 How many real solutions does $|x^2 - 8x| = 16$ have?

PART II: 10 MINUTES NYCIML Contest Three Spring 2013

S13B15 Find the area of the region enclosed by the graphs of $y = |4x| - 4$ and $y = -|4x| + 4$.

S13B16 In $\Delta ABC, BC = 36, \overline{DF} \parallel \overline{EG} \parallel \overline{BC}, \overline{ADEB}, \overline{AFGC}$, and the ratio of the areas of ΔADF to quadrilateral $DFGE$ to quadrilateral $EGBC = 4:5:7$. Find EG.

PART III: 10 MINUTES NYCIML Contest Three Spring 2013

S13B17 If $f(3-x) + (3-x)f(x) = 10$, find $f(4)$.

S13B18 The expression $\cot 40° + \tan 20°$ is equal to $\csc x°$. Find x.

ANSWERS

1. 1080
2. 3
3. 8
4. 27
5. -6
6. 40

SENIOR B DIVISION CONTEST NUMBER FOUR SPRING 2013

PART I: 10 MINUTES

S13B19 What is the units' digit of $\sum_{n=0}^{100} n!$?

S13B20 An equilateral triangle is circumscribed about a circle with radius of 12. Find the area of the triangle.

PART II: 10 MINUTES NYCIML Contest Four Spring 2013

S13B21 In a rectangular three dimensional coordinate system, the coordinates of the vertices of ΔPQR are $P\left(4\sqrt{2},0,0\right)$, $Q\left(0,4\sqrt{2},0\right)$ and $R\left(0,0,\sqrt{17}\right)$. Find the area of ΔPQR in simplest radical form.

S13B22 Express in simplest form: $\sum_{1}^{99} \frac{1}{\sqrt{n+1}+\sqrt{n}}$.

PART III: 10 MINUTES NYCIML Contest Four Spring 2013

S13B23 If quadrilateral $RSTU$ is inscribed in a circle with diameter $RU = 12$ and sides $RS = ST = 3$ and TU is expressed in simplest ${}^{p}/_{q}$ form, find $p+q$.

S13B24 If $\sin 10° \sin 40° \sin 50° = \frac{1}{k} \sin 20°$, find k.

ANSWERS

1. 4
2. $432\sqrt{3}$
3. $4\sqrt{33}$
4. 9
5. 23
6. 4

SENIOR B DIVISION CONTEST NUMBER FIVE SPRING 2013

PART I: 10 MINUTES

S13B25 How many pairs of positive integral solutions does $5x + 3y = 104$ have?

S13B26 The length of a diagonal of a cube is $24\sqrt{3}$. Find the sum of the lengths of all the edges of the cube.

PART II: 10 MINUTES NYCIML Contest Five Spring 2013

S13B27 Find all integral values of x such that $x^2 - 7x + 12$ is a positive prime number.

S13B28 A cube is inscribed in a sphere. If the surface area of the cube is 864, find the volume of the sphere expressed in simplest radical form.

PART III: 10 MINUTES NYCIML Contest Five Spring 2013

S13B29 In a rectangular prism, the area of the front face is 91 the area of the top face is 65 and the area of the left face is 35. Find the volume of the prism.

S13B30 Express in simplest form: $\sum_1^{50} \frac{1}{2n\cdot(2n+2)}$.

ANSWERS

1. 7
2. 288
3. 2 and 5
4. $864\pi\sqrt{3}$
5. 455
6. $^{25}/_{102}$

SENIOR B DIVISION CONTEST NUMBER ONE SPRING 2013

SOLUTIONS

1. **85**. Since $x^2 - y^2 = (x+y)(x-y)$, $77 = 11(x-y)$ and so $x - y = 7$. From $x + y = 11$ and $x - y = 7$ we may deduce that $x = 9$ and $y = 2$. Therefore, $x^2 + y^2 = 85$.

2. **18.** The given sum is an infinite geometric series with a ratio of 0.9. Therefore, it converges to $\frac{a_1}{1-r} = \frac{2 \cdot 0.9}{1-0.9} = \frac{1.8}{0.1} = 18$.

3. **18**. Use the angle bisector theorem: $\frac{3}{4} = \frac{AD}{42-AD} \rightarrow 126 - 3AD = 4AD \rightarrow 126 = 7AD \rightarrow AD = 18$.

4. **12**. Let the length of any side of the rhombus be x. The following proportion then results from $\Delta ABC \sim \Delta STC$. $\frac{20-x}{20} = \frac{x}{30} \rightarrow x = 12$.

5. **156**. The given statement means that there are integers a, b, and c such that $(*)\ 502 = an + r$, (**) $1021 = bn + r$, and (***) $2405 = cn + r$. Subtract the equations in pairs: (**) – (*): $519 = n(a-b) = 173 \cdot 3$, (***) – (**): $1384 = n(c-b) = 173 \cdot 8$, and (***) – (*): $1903 = n(c-a) = 173 \cdot 11$. So, $n = 173$. Since $502 = 173 \cdot 2 + 156, r = 156$.

6. **301.** The given triangle is a right triangle with right angle C. So, $\cos A = \frac{12}{13}$ and $\cos 2B = 2(\cos B)^2 - 1 = 2\left(\frac{5}{13}\right)^2 - 1 = -\frac{119}{169}$. Also, $\sin 3C = \sin 270° = -1$. Thus, the required sum of trigonometric functions is $-\frac{132}{169}$. Its absolute value is $\frac{132}{169}$. Therefore, $p = 132, q = 169$ and $p + q = 301$.

SENIOR B DIVISION CONTEST NUMBER TWO SPRING 2013

SOLUTIONS

7. **11**. Let $8x + 5 = 29$ Therefore $x = 3$ and $4x - 1 = 11$.

8. **12**. Let the length of a side of the equilateral triangle be x and let the length of a side of the regular hexagon be y. The regular hexagon can be partitioned into six equilateral triangles, each with a side of length y. Use the formula for the area of an equilateral triangle: $A = \frac{(side)^2\sqrt{3}}{4}$. Thus, $\frac{x^2\sqrt{3}/_4}{6y^2\sqrt{3}/_4} = 24$. So, $\frac{x^2}{6y^2} = 24 \rightarrow \frac{x^2}{y^2} = 144 \rightarrow \frac{x}{y} = 12$.

9. **41**. Let $RS = x$ and use the Pythagorean Theorem in ΔPTS: $9^2 + (x-1)^2 = x^2 \rightarrow x = 41$.

10. **2**. The value of $\sec\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}(1)\right) = \sec(30° - 90°) = \sec(-60°) = \frac{1}{\cos(-60°)} = \frac{1}{\cos(60°)} = 2$.

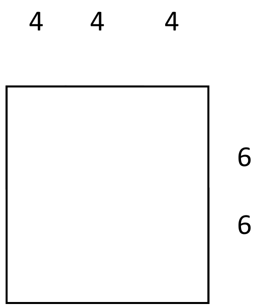
11. **$^2/_5$**. $f(g(x)) = \frac{3}{4x-2}$ and $g(f(x)) = \frac{12}{x-2}$. Set these equal and solve for x and $x = \frac{2}{5}$.

12. **$6\sqrt{5}$**. In rectangle $ABCD$, diagonal AC is drawn. Circle O is inscribed in right ΔADC and circle P is inscribed in right triangle . Circle $O$is tangent to $\overline{AD}$ at point E, to $\overline{CD}$ at point F, and to $\overline{AC}$ at point G. Circle P is tangent to $\overline{CB}$ at point H, to $\overline{AB}$ at point J, and to $\overline{AC}$ at point K. Point M is the midpoint of diagonal $\overline{AC}$. Since tangent segments to a circle from a common outside point are congruent, let $AE = AG = x, ED = DF = 18 - x$, and $FC = CG = 6 + x$. Use the Pythagorean Theorem in ΔACD to find $AC = 30$. So $AM = 15$ and $6 + x + x = 30 \rightarrow x = 12$. Since quadrilateral $DFOE$ is a square, $GM = 3$ and $DF = OG = 6$. Use the Pythagorean Theorem in ΔOMG: $3^2 + 6^2 = (OM)^2 \rightarrow OM = 3\sqrt{5}$. Use rotational symmetry to find $OP = 6\sqrt{5}$.

SENIOR B DIVISION CONTEST NUMBER THREE SPRING 2013

SOLUTIONS

13. **1080**.



The figure shows six four inch by six inch tiles comprising one square foot. The surface requires 180 square feet. This tiling requires $180 \cdot 6 = 1080$ tiles.

14. **3**. The given equation is equivalent to $x^2 - 8x = \pm 16 \rightarrow x^2 - 8x + 16 = 0$ or $x^2 - 8x - 16 = 0$. The solutions to these equations are $1 \pm \sqrt{2}$ and 4. So there are three solutions.

15. **8**. The graph of $y = |4x| - 4$ is a vee with a vertex at point $P\ (0, -4)$. The graph of $y = -|4x| + 4$ is an inverted vee with a vertex at point $Q\ (0,4)$. The graphs intersect at point $R\ (1,0)$ and point $S\ (-1,0)$. Quadrilateral $PQRS$ is a rhombus and its area is one-half the product of the lengths of its diagonals and $\frac{1}{2} \cdot 2 \cdot 8 = 8$.

16. **27**. Since $\overline{EG} \parallel \overline{BC}$, the ratio of the area of ΔAEG to the area of $\Delta ABC = \left(\frac{EG}{BC}\right)^2 = \frac{9}{16}$. Therefore, $\frac{EG}{BC} = \frac{3}{4}$ and $EG = 27$.

17. $-$**6**. If $x = 4$, the given equation becomes $f(-1) - f(4) = 10$.

If $x = -1$, the given equation becomes $f(4) + 4f(-1) = 10$.

Add the latter two equations to get $5f(-1) = 20$. So, $f(-1) = 4$. Therefore, $f(4) = -6$.

18. **40.** Use identities: $\cot 2x + \tan x = \frac{1}{\tan 2x} + \tan x = \frac{1-(\tan x)^2}{2\tan x} + \frac{2(\tan x)^2}{2\tan x} = \frac{1+(\tan x)^2}{2\tan x} = \frac{(\sec x)^2}{2\tan x} = \frac{1/(\cos x)^2}{2\sin x/\cos x} = \frac{1}{2\sin x\cos x} = \frac{1}{\sin 2x} = \csc 2x$. In our problem, $x = 20$. So, $2x = 40$.

SENIOR B DIVISION CONTEST NUMBER FOUR SPRING 2013

SOLUTIONS

19. **4**. $\sum_{n=0}^{100} n! = 0! + 1! + 2! + 3! + \cdots + 100!$ The units' digit of $0!$ is 1. The units' digit of $1!$ is 1. The units' digit of $2!$ is 2. The units' digit of $3!$ is 6. The units' digit of $4!$ is 4. The units' digit of $5!$ is 0. The units' digit of all of the rest of the summands is 0 too. The sum of these units' digits is 14. So the units' digit of the sum is 4.

20. $\mathbf{432\sqrt{3}}$. Let point O be the center of the circle and let point T be the point of tangency to $\overline{AC}$ in ΔABC. Draw radius $\overline{OT} \perp \overline{AC}$. Now, ΔOAT is $30° - 60° - 90°$ and $OT = 12$ so $AT = 12\sqrt{3}$ and $AC = 24\sqrt{3}$. So the area of the equilateral triangle is $A = \frac{s^2\sqrt{3}}{4} = 432\sqrt{3}$.

21. $\mathbf{4\sqrt{33}}$. Use the distance formula in three dimensions, $d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$ to get $PQ = 8$ and $PR = QR = 7$. Drop an altitude from vertex R to $\overline{PQ}$ in isosceles ΔPQR and use the Pythagorean Theorem: $4^2 + h^2 = 7^2 \rightarrow h = \sqrt{33}$. So, the area of ΔPQR is $4\sqrt{33}$.

22. **9**. Rationalize the denominator in the given expression to obtain $\sum_1^{99}\left(\sqrt{n+1} - \sqrt{n}\right) = \left(\sqrt{2} - \sqrt{1}\right) + \left(\sqrt{3} - \sqrt{2}\right) + \left(\sqrt{4} - \sqrt{3}\right) + \cdots + \left(\sqrt{100} - \sqrt{99}\right)$. This telescopes to $\sqrt{100} - \sqrt{1} = 9$.

23. **23**. Let point O be the center of the circle so radii $OR = OS = OT = OU = 6$. In circle O, chords RS and ST are congruent. So, the measures of central angles $m\measuredangle ROS = m\measuredangle SOT = x$ and $m\measuredangle TOU = 180 - 2x$. Then, use the law of cosines in ΔROS and ΔTOU: $3^2 = 6^2 + 6^2 - 2 \cdot 6 \cdot 6\cos x \rightarrow \cos x = \frac{7}{8}$. Also, $(TU)^2 = 6^2 + 6^2 - 2 \cdot 6 \cdot 6\cos(180 - 2x) = 36 + 36 + 72\cos 2x = 72(1 + \cos 2x) = 72 \cdot 2(\cos x)^2 = 144 \cdot \frac{49}{64}$. So, $TU = \frac{21}{2}, p = 21, q = 2$, and $p + q = 23$.

24. **4**. $\sin 10° \sin 40° \sin 50° = \cos 80° \sin 40° \cos 40° = \cos 80° \cdot \frac{1}{2}\sin 80° = \frac{1}{2} \cdot \frac{1}{2}\sin 160° = \frac{1}{4}\sin 20°$. So, $k = 4$.

SENIOR B DIVISION CONTEST NUMBER FIVE SPRING 2013

SOLUTIONS

25. **7**. One solution can be obtained by trial and error: $(1,33)$. Additional solutions are obtained by adding three to the x value and subtracting five from the y value until the y value is no longer positive. In this manner, we obtain: $(4,28), (7,23), (10,18), (13,13), (16,8)$, and $(19,3)$. So there are seven solutions.

26. **288**. Let the length of an edge of the cube be x. Then using the Pythagorean Theorem in three dimensions, $x^2 + x^2 + x^2 = \left(24\sqrt{3}\right)^2 \rightarrow 3x^2 = 3 \cdot 576 \rightarrow x^2 = 576 \rightarrow x = 24 \rightarrow 12x = 288$.

27. **2 and 5**. $x^2 - 7x + 12 = (x-4)(x-3)$. In order for this product to be prime exactly one of the factors must be ± 1. So, if $x - 3 = 1, x = 4$ and the product is 0 (not a prime).
If $x - 4 = 1, x = 5$ and the product is 2 (a prime).
If $x - 3 = -1, x = 2$ and the product is 2 (a prime).
Finally, if $x - 4 = -1, x = 3$ and the product is 0 (not a prime). The answer is 5 and 2.

28. $\boldsymbol{864\pi\sqrt{3}}$. If the length of the side of the cube is s and the surface area is $6s^2 = 864$, then $s = 12$. If the length of the diagonal of the cube is d, then $d^2 = 12^2 + 12^2 + 12^2$. So the length of the diagonal of the cube is $12\sqrt{3}$ and the length of the radius of the sphere is $6\sqrt{3}$ and the volume of the sphere is $V = \frac{4}{3}\pi r^3 = 864\pi\sqrt{3}$.

29. **455**. Call the three dimensions of the rectangular prism, x, y, and z. Then $xy = 91, yz = 65$, and $zx = 35$. Note that these three equations are symmetric in the three variables. So, multiply them to get $(xyz)^2 = 13 \cdot 7 \cdot 13 \cdot 5 \cdot 5 \cdot 7 = (13 \cdot 5 \cdot 7)^2$. So the volume of the prism, $xyz = 13 \cdot 5 \cdot 7 = 455$.

30. $\boldsymbol{{}^{25}/_{102}}$. We may re-write the given sum as $\sum_1^{50} \frac{1}{4}\left(\frac{1}{n} \cdot \frac{1}{n+1}\right) = \frac{1}{4}\sum_1^{50}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{4}\left[\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{50} - \frac{1}{51}\right)\right]$. Because of the telescoping nature of this sum, the sum may be re-written as $\frac{1}{4}\left(1 - \frac{1}{51}\right) = \frac{50}{204} = \frac{25}{102}$.