JUNIOR DIVISION **CONTEST NUMBER ONE** **SPRING 2013**

PART I: 10 minutes **NYCIML Contest One** **Spring 2013**

S13J1. Compute $2^{12} - (2^{11} + 2^{10} + 2^9)$.

S13J2. The base of a pyramid is an equilateral triangle whose sides have length 4, and the height of the pyramid is 6. The pyramid is divided into two solids by a plane parallel to the base that intersects three edges of the pyramid at their midpoints. Compute the volume of the larger of the two solids.

PART II: 10 minutes **NYCIML Contest One** **Spring 2013**

S13J3. If x and y are positive real numbers such that $x + 2y = 6$, find the maximum value of xy.

S13J4. Find the ordered triple (x, y, z) of positive integers with $x < y < z$ that satisfies $xyz + xy + yz + xz + x + y + z = 384$.

PART III: 10 minutes **NYCIML Contest One** **Spring 2013**

S13J5. In triangle ABC, the bisector of angle A meets $\overline{BC}$ at D, and the median from A meets $\overline{BC}$ at M. If $\frac{DM}{BC} = \frac{1}{16}$ and $AB < AC$, compute the ratio $\frac{AB}{AC}$.

S13J6. A teacher wishes to divide six students into four groups. How many such divisions are possible? (A group must have at least one member.)

JUNIOR DIVISION **CONTEST NUMBER TWO** **SPRING 2013**

PART I: 10 minutes **NYCIML Contest Two** **Spring 2013**

S13J7. Set $S = \{91, 92, \ldots, 98, 99\}$, and a, b, c, and d are distinct (different) elements of S. Compute the maximum value of $\frac{a+b}{c+d}$.

S13J8. There are 13 balls. Each ball is colored in one of ten different colors. Four of the balls are the same color. The colors of the other nine balls are different from each other and from the four balls that are colored the same. Four balls are chosen from the 13. How many different color selections are possible?

PART II: 10 minutes **NYCIML Contest Two** **Spring 2013**

S13J9. Find all three-digit numbers that are 25 times the sum of their digits.

S13J10. A spherical orange of radius 6 rests on a horizontal table. The orange is sliced one-fourth of the way from the table to the top of the orange so that the slice is horizontal. Find the area of the circular portion of the bottom portion of the orange that is thus exposed.

PART III: 10 minutes **NYCIML Contest Two** **Spring 2013**

S13J11. In triangle ABC, $m\angle C = 90$ and $m\angle B = 60$. The perimeter of triangle ABC is $3 - \sqrt{3}$. Compute AC.

S13J12. Compute the greatest real solution of $x^4 - 10x^2 + 1 = 0$.

JUNIOR DIVISION **CONTEST NUMBER THREE** **SPRING 2013**

PART I: 10 minutes **NYCIML Contest Three** **Spring 2013**

S13J13. Find the real number x such that $(x-10)^4 = (x+6)^4$.

S13J14. In triangle ABC, $AC = 21$, $BC = 29$, and $\overline{CM}$ is a median. Compute the number of possible integer values of CM.

PART II: 10 minutes **NYCIML Contest Three** **Spring 2013**

S13J15. A 10 by 10 square is divided into 100 unit squares by drawing lines parallel to its sides. Compute the number of sets consisting of two unit squares that are in neither the same row nor the same column.

S13J16. Find all ordered pairs (x, y) of positive integers that satisfy $xy + 5x + 6y = 383$.

PART III: 10 minutes **NYCIML Contest Three** **Spring 2013**

S13J17. In parallelogram $ABCD$, $\overline{DE}$ and $\overline{DF}$ are altitudes, $AB = 6$, $BC = 8$, and $DE = 4$. Compute BF.

S13J18. The radius of a cylindrical glass is 1, its height is 3, and the glass is $\frac{3}{4}$ full of water. The glass is tipped so that the highest point of the water touches the rim of the glass, which causes the surface of the water to form an ellipse. Find the ratio of the length of the major axis of the ellipse to the length of its minor axis.

JUNIOR DIVISION **CONTEST NUMBER ONE** **SPRING 2013**

SOLUTIONS

S13J1 $\mathbf{512.}$ Notice that $2^{12} - (2^{11} + 2^{10} + 2^9) = 2^{12} - 2^{11} - 2^{10} - 2^9 = 2^9(2^3 - 2^2 - 2 - 1) = 512(8 - 4 - 2 - 1) = 512$.

S13J2 $\mathbf{7\sqrt{3}.}$ The volume of a pyramid is $\frac{1}{3}$ the area of its base times its height. Use the formula $K = \frac{s^2\sqrt{3}}{4}$ for equilateral triangles to find that the area of the base is $4\sqrt{3}$. The volume is then $8\sqrt{3}$. One of the solids formed by the plane dividing the pyramid is a smaller pyramid that is similar to the original with a similarity ratio of $\frac{1}{2}$. The ratio of the volume of the smaller pyramid to that of the larger is therefore $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$. Thus, the volume of the small pyramid is $\sqrt{3}$ and the volume of the larger solid is $8\sqrt{3} - \sqrt{3} = 7\sqrt{3}$.

S13J3 $\mathbf{\frac{9}{2}.}$ Let $P = xy$. Then $P = (6 - 2y)y = -2y^2 + 6y$. The graph of P versus y is a parabola, and P attains its maximum when $y = -\frac{6}{2\cdot(-2)} = \frac{3}{2}$. Thus, the maximum value of P is $\left(\frac{3}{2}\right) \cdot 3 = \frac{9}{2}$.

Alternate solution: Use the Arithmetic Mean – Geometric Mean Inequality: $3 = \frac{x+2y}{2} \geq \sqrt{2xy}$ with equality when $x = 2y$. That is, when $x = 3$ and $y = \frac{3}{2}$. Thus, the maximum value of xy is $\left(\frac{3}{2}\right) \cdot 3 = \frac{9}{2}$.

S13J4 $\mathbf{(4, 6, 10).}$ The given equation is equivalent to $(x + 1)(y + 1)(z + 1) = 385 = 5 \cdot 7 \cdot 11$. The requested ordered triple is $(4, 6, 10)$.

S13J5 $\mathbf{\frac{7}{9}.}$ Let $DM = x$. Then $BC = 16x$, so $BM = MC = 8x$, and $BD = 7x$. Use the Angle-Bisector Theorem to conclude that $\frac{AB}{AC} = \frac{BD}{DC} = \frac{7}{9}$.

S13J6 $\mathbf{65.}$ There are two possibilities for the sizes of the four groups: 3, 1, 1, and 1 or 2, 2, 1, and 1. The number of divisions that correspond to the first case is $\binom{6}{3} = 20$ because the division is determined by the selection for the group of 3. The number of divisions that correspond to the second case is $\frac{1}{2} \cdot \binom{6}{2} \cdot \binom{4}{2} = 45$. This is because the division is determined by the selection of the two groups of two. There are $\binom{6}{2}$ selections for the first group, and then $\binom{4}{2}$ for the remaining group of two. But $\binom{6}{2} \cdot \binom{4}{2}$ counts each division twice. Thus, there are $20 + 45 = 65$ possible divisions.

JUNIOR DIVISION **CONTEST NUMBER TWO** **SPRING 2013**

SOLUTIONS

S13J7 $\mathbf{197.}$ The fraction will be a maximum when its numerator is maximum and its denominator is minimum. The maximum numerator is $98 + 99 = 197$ and the minimum denominator is 1, so the requested maximum is 197.

S13J8 $\mathbf{256.}$ A color selection consists of k balls from the four same colored balls and $4-k$ from the other nine, where $0 \le k \le 4$. The number of possible color selections is therefore $\binom{9}{0}+\binom{9}{1}+\binom{9}{2}+\binom{9}{3}+\binom{9}{4}=\frac{1}{2}\left(\binom{9}{0}\binom{9}{1}\binom{9}{2}+\cdots+\binom{9}{9}\right)=\frac{1}{2}\times 2^9=2^8=256$.

S13J9 $\mathbf{150, 225,}$ **and** $\mathbf{375.}$ Let n be an integer with the desired property, and let a, b, and c be the 100's, 10's and 1's digits of n, respectively. Then, $100a+10b+c=25a+25b+25c$, which is equivalent to $25a-5b=8c$. Thus 5 divides $8c$, so $c=0$ or $c=5$. In the case $c=0$, substitution yields $b=5a$, so $(a,b)=(1,5)$ When $c=5$, it follows that $5a=b+8$, so $(a,b)=(2,2)$ or $(3,7)$. Thus, the three possible values of n are 150, 225, and 375.

S13J10 $\boldsymbol{27\pi}.$ Let O be the center of the sphere, let P be the center of the circle, and let A be any point on the circle. Then triangle OPA is a right triangle with $OP=3$ and $OA=6$. Apply the Pythagorean Theorem to find that $PA=\sqrt{27}$. Thus, the area of the circle is 27π.

S13J11 $\mathbf{2\sqrt{3}-3.}$ Let $BC=s$. Then, $AC=s\sqrt{3}$ and $AB=2s$, and so $3-\sqrt{3}=s+s\sqrt{3}+2s=3s+s\sqrt{3}=s(3+\sqrt{3})$. Hence, $s=\frac{3-\sqrt{3}}{3+\sqrt{3}}=2-\sqrt{3}$, and $AC=s\sqrt{3}=2\sqrt{3}-3$.

S13J12 $\mathbf{\sqrt{2}+\sqrt{3}}$ **.** The given equation is equivalent to $x^4-2x^2+1=8x^2$, or $(x^2-1)^2=8x^2$. Thus, $x^2-1=\pm x\sqrt{8}$, so $x^2-x\sqrt{8}-1=0$ or $x^2+\sqrt{8}-1=0$. Use the quadratic formula to find that the four solutions are $\sqrt{2}\pm\sqrt{3}$ and $-\sqrt{2}\pm\sqrt{3}$, the greatest of which is $\sqrt{2}+\sqrt{3}$.

JUNIOR DIVISION CONTEST NUMBER THREE SPRING 2013

SOLUTIONS

S13J13 $\mathbf{2.}$ The given equation implies that $x - 10 = x + 6$ or $x - 10 = -(x + 6)$. The first of these has no solution, and the second has the solution $x = 2$.

S13J14 $\mathbf{20.}$ Extend $\overline{CM}$ its own length to point D. Then, $ACBD$ is a parallelogram, so $= 2CM$, and $AD = 29$. Apply the Triangle Inequality in triangle ACD to conclude that $29 - 21 < 2 \cdot CM < 21 + 29$. Thus, $4 < CM < 25$, and there are 20 possible integer values of *CM*.

S13J15 $\mathbf{4050.}$ There are 100 choices for the first square to be placed in the set, and then 81 choices for the second for a total of $100 \cdot 81 = 8100$ ordered pairs of squares that satisfy the given conditions. Because the order of selection of the two squares does not change the set itself there are $\frac{8100}{2} = 4050$ of the desired sets.

S13J16 $\mathbf{(1, 54), (53, 2).}$ The given equation is equivalent to $(x + 6)(y + 5) = 413 = 7 \cdot 59$. Then, $(x + 6, y + 5) = (1, 413)$, $(413, 1)$, $(7, 59)$, or $(59, 7)$. Only the last two ordered pairs yield positive values for x and y, so $(x, y) = (1, 54)$ or $(53, 2)$.

S13J17 $\mathbf{8 - 3\sqrt{3}.}$ Because triangle AED is similar to CFD, $\frac{8}{6} = \frac{AD}{DC} = \frac{DE}{DF} = \frac{4}{DF}$. Apply the Pythagorean Theorem in triangle CDF to find that $CF = 3\sqrt{3}$. Thus, $BF = 8 - 3\sqrt{3}$.

S13J18 $\mathbf{\frac{5}{4}.}$ The minor axis of the ellipse is a diameter of the cylinder. Let O be the center of the ellipse, let A be the point where the water touches the rim after the glass is tipped, and let B be the point that is directly beneath A at the same height as O before the glass is tipped. Then, $OB = 1$ and $AB = \frac{3}{4}$. Apply the Pythagorean Theorem in triangle OAB to find that $OA = \frac{5}{4}$. Thus, the requested ratio equals $\frac{OA}{OB} = \frac{5}{4}$.