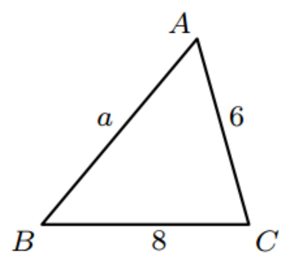
NYCIML Fall 2012 Sophomore-Freshman Division
Fall Contest Number 1

PART II: 10 minutes **NYCIML Contest One** **Fall 2012**

F12SF1. Compute the sum of all positive integer values that a can take in the triangle shown below.



F12SF2. If $x < 0$ and $x^2 + 3x = 10$, then compute $(x + 2)(x + 8)$.

PART II: 10 minutes **NYCIML Contest One** **Fall 2012**

F12SF3. Compute x that satisfies the following equation:

$$(x)(x^{3+\sqrt{3}})(x^{-3+\sqrt{5}})(x^{3-\sqrt{3}})(x^{-3-\sqrt{5}}) = 5$$

F12SF4. Compute the number of distinct ordered triplet of positive integers (x, y, z) such that $x + y + z = 80$.

PART II: 10 minutes **NYCIML Contest One** **Fall 2012**

F12SF5. If x, y, and z are real numbers, and if $|x| \leq 5$, $|y - 4| \leq 13$, and $|z - x| \leq 9$, compute the maximum value of $|x + y - z|$.

F12SF6. The two equations $x^5 + ax^3 + 2 = 0$ and $x^4 + ax^2 + 3 = 0$ (where a is a rational number) have a common root. Compute a.

NYCIML Fall 2012 Sophomore-Freshman Division
Fall Contest Number 2

PART I: 10 minutes **NYCIML Contest Two** **Fall 2012**

F12SF7. Given that a, b, and c are distinct, positive integers, compute the least possible value of $a + b^2 + c^3$.

F12SF8. Compute the sum of all possible solutions x to the equation $\left(x + \frac{4}{x}\right)^2 = 16$.

PART II: 10 minutes **NYCIML Contest Two** **Fall 2012**

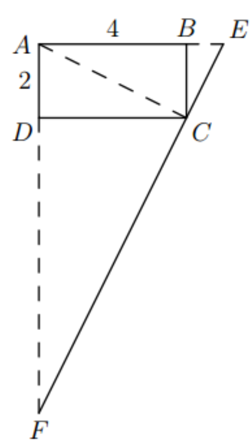
F12SF9. Compute the sum of all possible integer solutions x to the inequality $9 < (x-3)^2 < 64$.

F12SF10. There are six positive integers in a sequence $a_1, a_2, a_3, a_4, a_5, a_6$. Given that $a_2 = 3$, $a_5 = a_6$, and the sum of any four consecutive terms is 27, compute the sum of all six terms.

PART III: 10 minutes **NYCIML Contest Two** **Fall 2012**

F12SF11. There are two six-sided dice: one die has only odd numbers with two of each 1, 3, and 5, while the other one is a regular die. Compute the probability that the sum of the numbers on the tops of the two dice is a prime number.

F12SF12. In rectangle $ABCD$, we have $AB = 4$ and $AD = 2$. $\triangle AEF$ is constructed by drawing EF through C such that AC is perpendicular to EF as shown below. $CE + DF$ can be expressed as $\sqrt{a} + b$ where a and b are positive integers. Compute $a + b$.



NYCIML Fall 2012 Sophomore-Freshman Division
Fall Contest Number 3

PART I: 10 minutes **NYCIML Contest Three** **Fall 2012**

F12SF13. Compute the number of three-digit numbers with a 2 in the hundreds place and an odd number in the ones place.

F12SF14. Compute the largest possible perimeter of a rectangle with integer length and width and area 20.

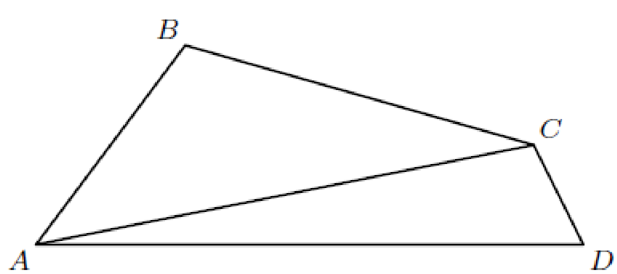
PART II: 10 minutes **NYCIML Contest Three** **Fall 2012**

F12SF15. Compute the largest number of consecutive integers whose sum is 2012.

F12SF16. There are two integers $x \geq 0$ for which the distance from $(6, x)$ to the origin is an integer. Compute the sum of the two integers.

PART III: 10 minutes **NYCIML Contest Three** **Fall 2012**

F12SF17. In the figure below, $AB = 5$, $BC = 8$, $CD = 2$, $AD = 13$, and AC is an integer. Compute AC.



F12SF18. Two real numbers a and b satisfy the two conditions

$$ab = 465$$
$$\frac{a^3 - b^3}{(a-b)^3} = \frac{36}{5}$$

Compute $a - b$.

Fall 2012 Sophomore-Freshman Division
Contest 1 SOLTUIONS

Problem 1. $\boxed{\textbf{88.}}$ From the triangle inequality theorem, we have the following three inequalities:

$$\begin{aligned} a+6 &> 8 \Longrightarrow a > 2 \\ a+8 &> 6 \Longrightarrow a > -2 \\ 6+8 &> a \Longrightarrow a > 14 \end{aligned}$$

Combining these three inequalities results in

$$2 < a < 14$$

Therefore, the sum of all the positive integers that a can take is 88.

Problem 2. $\boxed{\textbf{-9.}}$ We can factor the given equation to:

$$x^2 + 3x - 10 = 0 \Longrightarrow (x-2)(x+5) = 0$$

Since $x < 0$, $x = -5$. Therefore, $(x+2)(x+8) = (-3)(3) = -9$.

Problem 3. $\boxed{5.}$ After simplifying the give equation, we get:

$$\begin{aligned} (x)(x^6)(x^{-6}) &= 5 \\ x &= 5 \end{aligned}$$

Therefore, $x = 5$.

Problem 4. $\boxed{\textbf{3081.}}$ Let us denote one such point in 3-dimensional space as (x, y, z). If we assume that $x = 1$, then there are 78 points that satisfy the given conditions:

$$\begin{array}{ccc} (1, & 1 & ,78) \\ (1, & 2 & ,77) \\ (1, & 3 & ,76) \\ & \vdots & \\ (1, & 78 & ,1) \end{array}$$

Similarly, if $x = 2$, then there are 77 points. The pattern becomes clear: for $x = n$, there are $78 - (n-1)$ points whose coordinates sum to 80. The largest value x can take is $x = 78$ which results in only one valid point, $(78, 1, 1)$. Then we need to find the sum of the following arithmetic series:

$$78 + 77 + 76 + \cdots + 2 + 1 = \frac{(78+1)(78)}{2} = 3081$$

Problem 5. $\boxed{26.}$ We have the following three inequalities:

$$\begin{aligned} |x| &\leq 5 \Longrightarrow -5 \leq x \leq 5 && (1) \\ |y-4| &\leq 13 \Longrightarrow -9 \leq y \leq 17 && (2) \\ |z-x| &\leq 9 \Longrightarrow -9 \leq z - x \leq 9 && (3) \end{aligned}$$

Combining (1) and (2) results in

$$-14 \leq x + y \leq 22$$

Now to maximize $|x+y-z|$, we can maximize $x+y$ and minimize z (or maximize $-z$). The maximum value $x+y$ can take is 22 ($x=5$ and $y=17$) from the above inequality. Substituting $x=5$ into (3), we get: $-4 \leq z \leq 14$. Therefore, the smallest value z can take is -4. The maximum value of $|x+y-z|$ is then $|5+17-(-4)|=26$.

Problem 6. **-259/36.** We are given the following two equations:

$$\begin{aligned} f(x) &= x^5 + ax^3 + 2 \quad (1) \\ g(x) &= x^4 + ax^2 + 3 \quad (2) \end{aligned}$$

Let us denote the common root as b. It is important to note that all the roots of $g(x)$ are also roots of $xg(x)$. Since $f(b) = g(b) = bg(b) = 0$, b is also a root of $f(x) - xg(x)$:

$$f(x) - xg(x) = 2 - 3x = 0 \Longrightarrow x = \frac{2}{3}$$

Substituting $x = \frac{2}{3}$ into $g(x)$ and solving for a, we get $a = -\frac{259}{36}$.

Fall 2012 Sophomore-Freshman Division
Contest 2 SOLUTIONS

Problem 1. **8.** In order to minimize $a + b^2 + c^3$, we have to assign the smallest positive integer to c, the second smallest positive integer to b, and the third smallest positive integer to c. In other words, $a = 3$, $b = 2$, and $c = 1$ resulting in $3 + 2^2 + 1^3 = 8$.

Problem 2. **0.** Let us define

$$A = x + \frac{4}{x}$$

Then the given problem is

$$A^2 = 16 \Longrightarrow A = \pm 4$$

There are two cases to consider: $A = 4$ and $A = -4$. For the first case, we have

$$\begin{aligned} A = 4 \Longrightarrow x + \frac{4}{x} &= 4 \\ x^2 - 4x + 4 &= 0 \\ (x-2)^2 &= 0 \\ x &= 2 \end{aligned}$$

For $A = -4$, we have

$$\begin{aligned} A = -4 \Longrightarrow x - \frac{4}{x} &= 4 \\ x^2 + 4x + 4 &= 0 \\ (x+2)^2 &= 0 \\ x &= -2 \end{aligned}$$

Therefore, there are only two possible solutions: $x = \pm 2$. The sum of the two solutions is 0.

Problem 3. **24.** We can rewrite the given inequality as

$$3^2 < (x-3)^2 < 8^2$$

Then $(x-3)^2 = 4^2, 5^2, 6^2, \text{ or } 7^2$.

- $(x-3)^2 = 4^2 \Longrightarrow x = 7, -1$
- $(x-3)^2 = 5^2 \Longrightarrow x = 8, -2$
- $(x-3)^2 = 6^2 \Longrightarrow x = 9, -3$
- $(x-3)^2 = 7^2 \Longrightarrow x = 10, -4$

The sum of all possible solutions is then $7 + 8 + 9 + 10 - 1 - 2 - 3 - 4 = 24$.

Problem 4. **33.** Since $a_2 = 3$, we have

$$\begin{aligned} a_1 + a_3 + a_4 &= 24 \\ a_3 + a_4 + a_5 &= 24 \end{aligned}$$

From the two equations above, we have $a_1 = a_5 = a_6$. Since the sum of the last four terms is 27 and $a_3 + a_4 + a_5 = 24$,

$$a_3 + a_4 + a_5 + a_6 = 27 \Longrightarrow a_6 = 3$$

Overall, we have $a_1 = a_2 = a_5 = a_6 = 3$. The sum of all six terms is $27 + 6 = 33$.

Problem 5. $\boxed{\mathbf{4/9.}}$ Let a be the number on the top of the first die (the one with only odd numbers) and b be the number on the top of the regular die. There are 8 cases when $a+b$ is a prime number:

$$(a,b) \in \{(1,1),(1,2),(1,4),(1,6),(3,2),(3,4),(5,2),(5,6)\}$$

The probability for each case to occur is

$$\frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$$

Since there are 8 cases, the probability that $a+b$ is a prime number is

$$\frac{1}{18} \cdot 8 = \frac{4}{9}$$

Problem 6. $\boxed{\mathbf{13.}}$ First, we can compute AC using the Pythagorean theorem: $AC = 2\sqrt{5}$. Since $\triangle ADC$ and $\triangle CDF$ are similar, we have the following relationship:

$$\frac{AC}{CF} = \frac{AD}{DC} \Longrightarrow \frac{2\sqrt{5}}{CF} = \frac{2}{4} \Longrightarrow CF = 4\sqrt{5}$$

By the Pythagorean theorem, we can compute DF:

$$DF = \sqrt{CF^2 - DC^2} = 8$$

$\triangle BEC$ and $\triangle DCF$ are similar, and we can establish the following relationship:

$$\frac{CE}{CF} = \frac{BC}{DF} \Longrightarrow \frac{CE}{4\sqrt{5}} = \frac{2}{8} \Longrightarrow CE = \sqrt{5}$$

Therefore, $CE + DF = \sqrt{5} + 8$ where $a = 5$ and $b = 8$.

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Fall 2012 Sophomore-Freshman Division
Contest 3 SOLTUIONS

Problem 1. $\boxed{\textbf{50.}}$ There are ten three-digit numbers that start with 2 and end with an odd number. Since there are five odd numbers (1, 3, 5, 7, and 9), there are 50 three-digit numbers that meet the restrictions.

Problem 2. $\boxed{\textbf{42.}}$ Let a and b be the sides of the rectangle. Then we have $ab = 20$. Since a and b are integers, there are only three possible pairs for a and b: (1, 20), (4, 5), and (2, 10). It is clear that (1, 20) results in the largest perimeter: $2(1 + 20) = 42$.

Problem 3. $\boxed{4024.}$ We can express 2012 as a sum of consecutive integers:

$$(-2011) + (-2010) + (-2009) + \cdots + (2009) + (2010) + (2011) + (2012) = 2012$$

Therefore, there are 4024 consecutive integers that sum to 2012.

Problem 4. $\boxed{8.}$ Let a be the distance between $(6, x)$ and $(0, 0)$. Then we have the following equation:

$$\begin{aligned} \sqrt{36 + x^2} &= a \\ 36 + x^2 &= a^2 \\ a^2 - x^2 &= 36 \\ (a + x)(a - x) &= 36 \end{aligned}$$

In other words, $(a + x)$ and $(a - x)$ are two factors of 36 which has five distinct pairs of factors: (1, 36), (2, 18), (3, 12), (4, 9), and (6, 6). Since both a and x must be integers, only (2, 18) and (6, 6) are valid.

- $a + x = 18$ and $a - x = 2$
$$a = 10 \text{ and } x = 8$$
- $a + x = 6$ and $a - x = 6$
$$a = 6 \text{ and } x = 0$$

Therefore, the two possible integers that x can take are 0 and 8.

Problem 5. $\boxed{\textbf{12.}}$ By the triangle inequality, we have the following inequalities:

$$\begin{aligned} AB + BC &> AC \Longrightarrow 13 > AC \\ AC + 2 &> AD \Longrightarrow AC > 11 \end{aligned}$$

Since $11 < AC < 13$, $AC = 12$.

Problem 6. $\boxed{\textbf{15.}}$ Applying the hint on the second condition, we get

$$\frac{a^3 - b^3}{(a - b)^3} = \frac{a^2 + ab + b^2}{(a - b)^2}$$

Now let us define $A = a - b$ and rewrite the above equation:

$$\frac{a^2 + ab + b^2}{(a - b)^2} = \frac{A^2 + 3ab}{A^2} = 1 + \frac{3ab}{A^2} = \frac{36}{5} \longrightarrow \frac{3ab}{A^2} = \frac{31}{5}$$

Then solving for A, we get

$$A = \sqrt{\frac{15}{31}ab}$$

Substituting the first condition $ab = 465 = 31 \cdot 3 \cdot 5$ into the above equation results in

$$A = a - b = \sqrt{15^2} = 15$$