New York City Interscholastic Math League
Senior B Division   Contest Number 1

Part I   Fall 2012   Contest 1   Time: 10 Minutes

F12B7 Compute the number of positive integers less than or equal to 1,000,000 that are either squares or cubes.

F12B8 In right triangle ABC, $m\angle C = 90^\circ$, $AC = 35$, and $BC = 12$. Compute the length of the radius of the inscribed circle.

Part II   Fall 2012   Contest 1   Time: 10 Minutes

F12B9 Find all ordered pairs of positive integers (x, y) such that $x \leq y$ and $xy = 7(x + y)$.

F12B10 The face diagonals of a rectangular prism have lengths 7, 8, and 9. If the length of the space diagonal of the prism is $\sqrt{k}$, compute k.

Part III   Fall 2012   Contest 1   Time: 10 Minutes

F12B11 If $\log_c a + \log_{c^2} a = 12$, then $a = c^k$. Compute k.

F12B12 In triangle KLM, the length of median $\overline{KP}$ is 15, the length of median $\overline{MQ}$ is 36, and the length of side $\overline{MK}$ is 26. Compute the area of KLM.

New York City Interscholastic Math League
Senior B Division   Contest Number 2

Part I   Fall 2012   Contest 2   Time: 10 Minutes

F12B7 From the fourth power of 125, subtract the third power of 125 and then divide the difference by 124. Find the cube root of the quotient.

F12B8 In triangle ABC, the lengths of sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are 5, 12, and 13, respectively. The angle bisector of B intersects $\overline{AC}$ at point D. Compute the length of segment $\overline{BD}$.

Part II   Fall 2012   Contest 2   Time: 10 Minutes

F12B9 If $x + \frac{2}{x} = 4$, find the value of $x^3 + \frac{8}{x^3}$.

F12B10 Find all values of θ on $[0, \pi]$ such that $(2^4)^{\cos^2\theta} + (2^4)^{\sin^2\theta} = 10$.

Part III   Fall 2012   Contest 2   Time: 10 Minutes

F12B11 In isosceles trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$, $AB = 40$, $CD = 10$, and $BC = AD = 20$. Points E and F are on $\overline{BC}$ and $\overline{AD}$ respectively, and $\overline{EF}$ is drawn parallel to $\overline{AB}$. The two smaller trapezoids thus formed are equal in perimeter. Find EF.

F12B12 Four fair dice are rolled. In simplest form, compute the probability that the product of the four resulting numbers is 16.

New York City Interscholastic Math League

Senior B Division   Contest Number 3

Part I   Fall 2012   Contest 3   Time: 10 Minutes

F12B13 The line given by the equation $y-2x-3=0$ is reflected over the y-axis, and then its image is reflected over the x-axis. Write the equation of the resulting final image in the form $y=mx+b$.

F12B14 In a triangle with sides of length 20, 30, and 40, how many times longer is the longest altitude than the shortest altitude?

Part II   Fall 2012   Contest 3   Time: 10 Minutes

F12B15 The average of n numbers is 30. If 1 is added to the first number, 2 is added to the second number, and so on until n is added to the n^{th} number, the average doubles to 60. Determine the value of n.

F12B16 In $\triangle PQR$, $\angle P$ is right, $PQ=\sqrt{63}$, and QR and PR are both integers. Find and list all possible ordered pairs of lengths $(QR,\ PR)$.

Part III   Fall 2012   Contest 3   Time: 10 Minutes

F12B17 Compute the area of the bounded region of the xy-plane consisting of all points whose coordinates satisfy the following inequalities: $y \leq 2x+1$, $y \geq 0$, $x+y \leq 4$, and $x \geq 0$.

F12B18 Find all real values of x for which $x^4-7x^3+11x^2+7x-12=0$.

New York City Interscholastic Math League
Senior B Division   Contest Number 4

Part I   Fall 2012   Contest 4   Time: 10 Minutes

F12B19 Find the sum of all possible values of x less than or equal to 75 such that $7^x - 3^x$ is a multiple of ten.

F12B20 For a raffle with two prizes, twenty-five tickets are sold. Two winning tickets will be selected without replacement. If you buy two tickets, what is the probability that you will win at least one prize?

Part II   Fall 2012   Contest 4   Time: 10 Minutes

F12B21 The first two terms of a geometric sequence are $\sqrt[3]{2}$ and $\sqrt[4]{2}$. The next term is $\sqrt[a]{2}$. Compute a.

F12B22 Cube $RSTVWXYZ$ has side length 12. Let point A be the center of face $WXYZ$. Find AR.

Part III   Fall 2012   Contest 4   Time: 10 Minutes

F12B23 Triangle RST is inscribed in circle O, with diameter $\overline{ROS}$. Through point O, a line perpendicular to diamater $\overline{ROS}$ intersects chord $\overline{RT}$ at point U. If $OU = 10$, and $m\angle OUR = 60°$, find TU.

F12B24 Find all ordered pairs (x, y) of positive integers that satisfy

$$(x + 3y - 2)^2 = x^2 + 9y^2 + 4$$

New York City Interscholastic Math League
Senior B Division — Contest Number 5

Part I — Fall 2012 — Contest 5 — Time: 10 Minutes

F12B25 Determine the value of $\log_{\sqrt{5}} \sqrt[3]{25}$.

F12B26 Find all ordered pairs of real numbers (x, y) such that $y - x = |x + y|$ and $3x + y = 12$.

Part II — Fall 2012 — Contest 5 — Time: 10 Minutes

F12B27 Two numbers have a geometric mean of 8 and an arithmetic mean of 7.5. Find the sum of the squares of the numbers.

F12B28 Evaluate

$$\sum_{x=41}^{49} \cos^2 x^\circ$$

Part III — Fall 2012 — Contest 5 — Time: 10 Minutes

F12B29 The point with coordinates $(4, -3)$ is reflected across the line whose equation is $y = \frac{1}{2}x$. Determine the coordinates of its image.

F12B30 If $f(x)$ is a function such that for each real number x, $f(-x) + 3f(x) = \tan x$, find the value of $f(\frac{\pi}{4})$.

New York City Interscholastic Math League
Senior B Division — Contest Number 1 Solutions

F12B1. **Answer : 1090**
There are 1000 perfect squares and 100 perfect cubes less than or equal to one million. However, there are 10 such perfect cubes that are also perfect squares; that is, numbers less than or equal to one million that are perfect sixth powers. So, the total number of unique cubes and squares is $1000 + 100 - 10 = \boxed{1090}$.

F12B2. **Answer : 5**
The Pythagorean Theorem yields $AB = 37$. Let the center of the inscribed circle be point O, and let the length of its radius be r. Let circle O be tangent to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ at points D, E, and F respectively. Since quadrilateral $ODCE$ is a square, $CD = CE = r$. Now, $BD = BF = 12 - r$ and $AE = AF = 35 - r$. Since $BF + FA = BA$, $12 - r + 35 - r = 37$. Therefore, $r = \boxed{5}$.

F12B3. **Answer : $(\mathbf{14}, \mathbf{14})$ and $(\mathbf{8}, \mathbf{56})$**
We are given that $x \le y$ and $xy = 7(x + y)$. Rearranging the second relation, we see that $xy - 7x - 7y = 0$. Then, $(x - 7)(y - 7) = xy - 7x - 7y + 49 = 49$. This equation yields three possibilities: $x - 7 = 7$ and $y - 7 = 7$, $x - 7 = 1$ and $y - 7 = 49$, or $x - 7 = 49$ and $y - 7 = 1$. The resulting values for (x, y) are $(14, 14)$, $(8, 56)$, and $(56, 8)$. The third is rejected due to the first condition, leaving $\boxed{(14, 14) \text{ and } (8, 56)}$.

F12B4. **Answer : 97**
Let the edge lengths be a, b, and c. Then, $a^2 + b^2 = 49$, $b^2 + c^2 = 64$, and $c^2 + a^2 = 81$. Adding, we get $2(a^2 + b^2 + c^2) = 194$. So, $a^2 + b^2 + c^2 = 97$. Therefore, the length of the space diagonal is $\sqrt{97}$. Therefore, $k = \boxed{97}$.

F12B5. **Answer : 8**
Rewrite the given equation as $\frac{\log a}{\log c} + \frac{\log a}{2 \log c} = 12$. Then, $\log_c a = \frac{2}{3} \cdot 12 = 8$, which means $c^8 = a$. So, $k = \boxed{8}$.

F12B6. **Answer : 360**
Let the point of intersection of $\overline{MQ}$ and $\overline{KP}$ be T. Since the medians of a triangle intersect at the centroid in a $2:1$ ratio, $MT = 24$. $TQ = 12$, $KT = 10$, and $TP = 5$. Since KTM is a 5-12-13 triangle, $\angle KTM$ is a right angle and the area of KTM is 120. Also, the areas of MTP and KTQ are both 60. Since a median separates a triangle into two triangles of equal area, the area of KTQ is equal to the area of LQT, which must also then be 60. Similarly, the area of MTP is equal to the area of LTP. So, the total area of KLM is $\boxed{360}$.

F12B7. **Answer : 125**

$$\sqrt[3]{\frac{125^4 - 125^3}{124}} = \sqrt[3]{\frac{125^3(125-1)}{124}} = \sqrt[3]{125^3} = \boxed{125}.$$

F12B8. **Answer : 79**

Let $AD = x$. Then, $CD = 13 - x$. Use the angle bisector theorem to get $\frac{5}{12} = \frac{x}{13-x}$. Solving this equation, we see that $x = \frac{65}{17}$. Use the law of sines in $\triangle ADB$ to get $\frac{\sin A}{BD} = \frac{\sin 45^\circ}{x}$. It follows that $\frac{12/13}{BD} = \frac{\sqrt{2}/2}{65/17}$, and $BD = \boxed{\dfrac{60\sqrt{2}}{17}}$.

F12B9. **Answer : 40**

$(x + \frac{2}{x})^3 = x^3 + 6x + \frac{12}{x} + \frac{8}{x^3} = 64$. Thus, $x^3 + \frac{8}{x^3} = 64 - 6(x + \frac{2}{x}) = 64 - 6 \cdot 4 = \boxed{40}$.

F12B10. **Answer :** $\boldsymbol{\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}}$

$(2^4)^{\cos^2\theta} + (2^4)^{\sin^2\theta} = 10$. So, $2^{4\cos^2\theta} + 2^{4\sin^2\theta} = 10$. It follows that $2^{4\cos^2\theta} + 2^{4-4\cos^2\theta} = 10$, which means $2^{4\cos^2\theta} + \frac{2^4}{2^{4\cos^2\theta}} = 10$. This is a quadratic in $2^{4\cos^2\theta}$. Solving, we find that $2^{4\cos^2\theta}$ must be 2 or 8. Therefore, $4\cos^2\theta = 1$ or 3. Solving for θ, we get $\boxed{\dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{5\pi}{6}}$.

F12B11. **Answer :** $\boldsymbol{\frac{145}{4}}$

Let $DF = x$. Since the perimeters of the two smaller trapezoids are equal, $10 + 2x = 40 + 2(20 - x)$. So, $x = DF = CF = \frac{35}{2}$, and $AF = BE = \frac{5}{2}$. Let diagonal $\overline{AC}$ intersect $\overline{EF}$ at point G. From $\triangle AFG \sim \triangle ADC$, we get $\frac{5/2}{20} = \frac{FG}{10}$ and $FG = \frac{5}{4}$. From $\triangle CGE \sim \triangle CAB$, we get $\frac{35/2}{20} = \frac{GE}{40}$ and $GE = 35$. Thus, $FE = FG + GE = \boxed{\dfrac{145}{4}}$.

F12B12. **Answer :** $\boldsymbol{\frac{19}{1296}}$

There are three distinct ways we can select four numbers whose product is 16: $\{1,1,4,4\}$, $\{1,2,2,4\}$, and $\{2,2,2,2\}$. There are $\frac{4!}{2!2!} = 6$ ways we can order the first, $\frac{4!}{2!} = 12$ ways we can order the second, and 1 way to order the third. Together they make up a total of 19 arrangements that produce a product of 16. Overall, we can roll the dice in $6^4 = 1296$ different ways. The probability of rolling a product of 16 is therefore $\boxed{\dfrac{19}{1296}}$.

F12B13. **Answer** : $\mathbf{y = 2x - 3}$
Rewrite the given equation as $y = 2x + 3$. Reflecting over the y-axis, we obtain $y = -2x + 3$ for the first image. Then, we reflect over the x-axis to obtain the final image $\boxed{y = 2x - 3}$.

F12B14. **Answer** : **2**
Let h_l be the length of the longest altitude of the triangle, which is the altitude to the side of length 20. Let h_s be the length of the shortest altitude of the triangle, which is the altitude to the side of length 40. The area of the triangle may be expressed either as $\frac{20h_l}{2}$ or $\frac{40h_s}{2}$. These expressions must be equal, so $\frac{h_l}{h_s} = \boxed{2}$.

F12B15. **Answer** : **59**
The sum of the numbers in the modified sequence can be written as $30n + (1 + 2 + 3 + \ldots + n)$. It can also be written as $60n$. Setting these two expressions equal, we obtain a quadratic in n:

$$30n + \frac{n(n+1)}{2} = 60n \Rightarrow \frac{n(n+1)}{2} = 30n \Rightarrow n^2 + n = 60n \Rightarrow n(n-59) = 0$$

Discarding the case where $n = 0$, we find that $n = \boxed{59}$.

F12B16. **Answer** : $(\mathbf{32}, \mathbf{31}), (\mathbf{12}, \mathbf{9}), (\mathbf{8}, \mathbf{1})$
From the Pythagorean Theorem, we know $QR^2 = 63 + PR^2$. Rearranging and factoring, we obtain $(QR + PR)(QR - PR) = 63 \cdot 1 = 21 \cdot 3 = 9 \cdot 7$. This yields three possible systems of equations: $QR + PR = 63$ and $QR - PR = 1$, or $QR + PR = 21$ and $QR - PR = 3$, or $QR + PR = 9$ and $QR - PR = 7$. The solutions are $(QR, PR) = \boxed{(32, 31), (12, 9), (8, 1)}$.

F12B17. **Answer** : $\mathbf{\frac{13}{2}}$
The boundary lines $y = 2x + 1$ and $x + y = 4$ intersect at point $(1, 3)$. The required area is the difference of the area of two triangular regions: $\frac{1}{2} \cdot \frac{9}{2} \cdot 3 - \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{27-1}{4} = \boxed{\frac{13}{2}}$.

F12B18. **Answer** : $\mathbf{-1, 1, 3, 4}$

$$\begin{aligned}
x^4 - 7x^3 + 11x^2 + 7x - 12 &= 0 \\
x^4 + 11x^2 - 12 - 7x^3 + 7x &= 0 \\
(x^2 + 12)(x^2 - 1) - 7x(x^2 - 1) &= 0 \\
(x^2 - 7x + 12)(x^2 - 1) &= 0 \\
(x - 4)(x - 3)(x - 1)(x + 1) &= 0 \\
x &= \boxed{-1, 1, 3, 4}
\end{aligned}$$

Senior B Division   Contest Number 4 Solutions

F12B19. **Answer : 1406**
Taking the units digit of 7^x for $x = 1, 2, 3, \ldots$ gives us $7, 9, 3, 1, 7, 9, 3, 1, \ldots$.
Taking the units digit of 3^x for $x = 1, 2, 3, \ldots$ gives us $3, 9, 7, 1, 3, 9, 7, 1, \ldots$.
So, $7^x - 3^x$ is a multiple of 10 when $x \equiv 0$ or $2 \pmod 4$. Summing $2+4+6+\cdots+74$ we obtain $\boxed{1406}$.

F12B20. **Answer : $\frac{47}{300}$**
The probability that you do not win either prize is given by $\frac{23}{25} \cdot \frac{22}{24} = \frac{253}{300}$. So, the probability that you win at least one prize must be $1 - \frac{253}{300} = \boxed{\dfrac{47}{300}}$.

F12B21. **Answer : 6**
Let the ratio of the geometric sequence be r. From the first two terms, we know $2^{1/3} \cdot r = 2^{1/4}$, or $r = 2^{1/4} \cdot 2^{-1/3} = 2^{-1/12}$. The third term must be $2^{1/4} \cdot 2^{-1/12} = 2^{1/6}$. So, $a = \boxed{6}$.

F12B22. **Answer : $6\sqrt{6}$**
Orient the cube so that A is at the origin, face $WXYZ$ lies in the plane $z = 0$, and point R has all positive coordinates. Then, $R = (6, 6, 12)$. Since A is at the origin, $AR = \sqrt{6^2 + 6^2 + 12^2} = \boxed{6\sqrt{6}}$.

F12B23. **Answer : 10**
Since $\overline{OU}$ is perpendicular to $\overline{ROS}$, $\angle ROU$ is right. This means $\triangle ROU$ is a 30-60-90 right triangle. Using $OU = 10$, we find $RU = 20$ and $RO = 10\sqrt{3}$. Diameter $\overline{RS}$ must have twice the length of RO, so $RS = 20\sqrt{3}$. $\triangle RTS$ is inscribed in a semicircle, so it is also right; in fact, it is also a 30-60-90 triangle. Using $RS = 20\sqrt{3}$ we find that $RT = 30$. $TU = RT - RU = 30 - 20 = \boxed{10}$.

F12B24. **Answer : $(\mathbf{3}, \mathbf{2}), (\mathbf{6}, \mathbf{1})$**

$$\begin{aligned}
(x + 3y - 2)^2 &= x^2 + 9y^2 + 4 \\
x^2 + 6xy - 4x + 9y^2 - 12y + 4 &= x^2 + 9y^2 + 4 \\
6xy - 4x - 12y &= 0 \\
3xy - 2x - 6y &= 0 \\
2x + 6y - 3xy - 4 &= -4 \\
(x - 2)(2 - 3y) &= -4
\end{aligned}$$

Since the solutions must be integers, the $(x - 2)$ factor must be $1, 2, 4, -1, -2, \text{or} -4$. Checking each possibility, we find that the only choices that produce positive integers for both x and y are $(x - 2) = 1$ and $(x - 2) = 4$, yielding $\boxed{(3, 2) \text{ and } (6, 1)}$.

Senior B Division Contest Number 5 Solutions

F12B25. **Answer** : $\mathbf{\frac{4}{3}}$

Let $x = \log_{\sqrt{5}} \sqrt[3]{25}$. Then, $(\sqrt{5})^x = \sqrt[3]{25}$, or $(5^{1/2})^x = (5^2)^{1/3}$. This means $\frac{1}{2}x = \frac{2}{3}$, or $x = \boxed{\frac{4}{3}}$.

F12B26. **Answer** : $(\mathbf{0}, \mathbf{12})$

We consider two separate cases: one where $x+y$ is positive, and one where $x+y$ is negative. In the first, $y - x = |x+y| = x+y$, implying $x = 0$. Using the fact that $3x + y = 12$ we obtain the solution $(0, 12)$. In the second case, $y - x = |x+y| = -x-y$, implying $y = 0$. Using the fact that $3x + y = 12$ we obtain the solution $(4, 0)$. However, we assumed that $x + y$ was negative to obtain this solution, while $4 + 0$ is not. So, we discard $(4, 0)$, leaving just $\boxed{(0, 12)}$.

F12B27. **Answer** : **97**

Let the numbers be x and y. We are given $\sqrt{xy} = 8$ and $\frac{x+y}{2} = 7.5$. Rewriting the second equation as $x + y = 15$ and squaring both sides, we obtain $x^2 + y^2 + 2xy = 225$. But $xy = 64$, so $x^2 + y^2 = 225 - 2 \cdot 64 = \boxed{97}$.

F12B28. **Answer** : $\mathbf{\frac{9}{2}}$

Observe $\cos^2 41° + \cos^2 49° = \cos^2 41° + \sin^2 41° = 1$. Similarly $\cos^2 42° + \cos^2 48° = \cos^2 43° + \cos^2 47° = \cos^2 44° + \cos^2 46° = 1$. Finally $\cos^2 45° = \frac{1}{2}$. So, the overall sum is $1+1+1+1+\frac{1}{2} = \boxed{\frac{9}{2}}$.

F12B29. **Answer** : $(\mathbf{0}, \mathbf{5})$

Through point $P(4, -3)$ draw a line perpendicular to the line $y = \frac{1}{2}x$. Let the lines intersect at point $Q(a, \frac{a}{2})$. Since $\overline{PQ}$ is perpendicular to a line of slope $\frac{1}{2}$, its slope is -2. So, $\frac{-3-\frac{a}{2}}{4-a} = -2$, implying $a = 2$. To get from $P(4, -3)$ to $Q(2, 1)$ we go left 2 and up 4. Continuing from $(2, 1)$ to reach the image of P, we obtain $(2-2, 1+4) = \boxed{(0, 5)}$.

F12B30. **Answer** : $\mathbf{\frac{1}{2}}$

Plugging $x = \frac{\pi}{4}$ and $x = -\frac{\pi}{4}$ into the given relation produces a system of two equations: $f(-\frac{\pi}{4}) + 3f(\frac{\pi}{4}) = \tan(\frac{\pi}{4}) = 1$ and $f(\frac{\pi}{4}) + 3f(-\frac{\pi}{4}) = \tan(-\frac{\pi}{4}) = -1$. Solving, we obtain $f(\frac{\pi}{4}) = \boxed{\frac{1}{2}}$.