JUNIOR DIVISION **CONTEST NUMBER ONE** **FALL 2012**
PART I: 10 minutes **NYCIML Contest One** **Fall 2012**

F11J1. A 6 by 8 rectangle has its vertices on a circle. Compute the area of the circle.

F11J2. If $\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}} = \frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{a}$, where a, b, and c are positive integers, compute (a, b, c).

PART II: 10 minutes **NYCIML Contest One** **Fall 2012**

F11J3. Let $n = 1{,}000{,}000{,}000$. Compute the number of times 0 appears as a digit in n^n.

F11J4. In ΔABC, $\angle C$ is a right angle, $AC = 5$, and $BC = 4$. Point P is on $\overline{AB}$, and line segments are drawn from P perpendicular to $\overline{AC}$ and $\overline{BC}$, meeting them at Q and R, respectively. Compute the minimum value of QR.

PART III: 10 minutes **NYCIML Contest One** **Fall 2012**

F11J5. Solve for n: $\sqrt[3]{24} + \sqrt[3]{81} = \sqrt[3]{n}$.

F11J6. Find the number of positive integers less than 1000 that contain at least one odd digit and at least one even digit.

JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 2012**
PART I: 10 minutes **NYCIML Contest Two** **Fall 2012**

F12J7. In rectangle $ABCD$, $AB = 4$, and E is chosen on AD so that $AE = 3.12$ and $ED = 6.88$. Compute the sum of the areas of triangles ABE and CDE.

F12J8. The IML Board of Directors picks a committee from its 10 members, and then one member of the committee is selected as a chairperson. Committees may consist of one or more members, and two committees are considered different if they have different chairpersons, even if all their members are the same. Compute the number of different committees that can be selected.

PART II: 10 minutes **NYCIML Contest Two** **Fall 2012**

F12J9. Find the smallest positive integer that leaves a remainder of 18 when divided by 19 and a remainder of 19 when divided by 20.

F12J10. Let $f(n)$ be a function defined as follows: $f(n) = \begin{cases} \frac{n^2-2}{n^3} & if\ n \neq 0 \\ 0 & if\ n = 0 \end{cases}$ Compute

$$\sum_{n=-5}^{6} f(n)$$

PART III: 10 minutes **NYCIML Contest Two** **Fall 2012**

F12J11. Compute the least positive integer n for which there exist an integer m such that $50^5 \times n^2 = m^3$.

F12J12. In square $ABCD$, E is on $\overline{BC}$ and F is on $\overline{CD}$ so that $\overline{AE} \perp \overline{EF}$. If $AE = 4$ and $EF = 3$, compute the area of $ABCD$.

JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 2012**
PART I: 10 minutes **NYCIML Contest Three** **Fall 2012**

F12J13. The perimeter of a rectangle with integer sides is 98. Compute the maximum area.

F12J14. Trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{CD}$ is inscribed in a circle. $AB = 6$, $CD = 9$ and $BD = 11$. Find AD.

PART II: 10 minutes **NYCIML Contest Three** **Fall 2012**

F12J15. In triangle ABC, $m\angle A = 60$, $m\angle B = 75$, and $BC = 6$. Compute AB.

F12J16. Set $A = \{3, 5, 8, 13, 21\}$, and set $B = \{4, 6, 9, 14, 22\}$. Compute the sum of the 25 possible products xy, where x is an element of A and y is an element of B.

PART III: 10 minutes **NYCIML Contest Three** **Fall 2012**

F12J17. Compute the number of divisors of 6^6 that are not divisible by 6.

F12J18. Given that $P(x)$ is a polynomial with nonnegative integer coefficients, and that $P(1) = 4$ and $P(5) = 1276$, compute $P(10)$.

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F12J1 $\mathbf{25\pi.}$ Because each edge of the rectangle is a right angle inscribed in a circle, its diagonal must also be a diameter of the circle. Use the Pythagorean Theorem to find that the diagonal is 10, and that therefore the radius of the circle is 5. Thus the area of the circle is 25π.

F12J2 $\mathbf{(12, 18, 30).}$ Notice that

$$\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}} = \frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}\frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}-\sqrt{5}} = \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{\left(\sqrt{2}+\sqrt{3}\right)^2-5} = \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2\sqrt{6}}$$

$$= \frac{\sqrt{12}+\sqrt{18}-\sqrt{30}}{12}$$

Thus $(a, b, c) = (12, 18, 30)$.

F12J3 $\mathbf{9\times10^9}$ *or* $\mathbf{9{,}000{,}000{,}000.}$ Notice that $n^n = (10^9)^{(10^9)} = 10^{9\times10^9}$. Thus n^n is written as a 1 followed by 9×10^9 0's.

F12J4 $\mathbf{\frac{20\sqrt{41}}{41}.}$ Because $CRPQ$ is a rectangle, it follows that $QR = CP$. But $QR = CP$ is a minimum when $\overline{CP} \perp \overline{AB}$, that is when $\overline{CP}$ is the altitude to the hypotenuse of ΔABC. Thus CP is a minimum when $AB \times CP = 2|ABC| = AC \times BC$. The minimum value of QR is therefore $\frac{20\sqrt{41}}{41}$.

F12J5 $\mathbf{375.}$ The given equation implies that $\sqrt[3]{n} = \sqrt[3]{8}\sqrt[3]{3} + \sqrt[3]{27}\sqrt[3]{3} = 2\sqrt[3]{3} + 3\sqrt[3]{3} = 5\sqrt[3]{3}$, so $n = 5^3 \times 3 = 375$.

F12J6 $\mathbf{720.}$ Count the complement. The complement of the given set consists of integers that contain no odd digits or integers that contain no even digits. All nine one-digit integers are in the complement. The number of two-digit integers that contain no odd digits is $4 \times 5 = 20$ because the leading digit cannot be 0, while the number of two-digit integers that contain no even digits is $5 \times 5 = 25$. Similarly, the number of three-digit integers that contain no odd digits is $4 \times 5 \times 5 = 100$, and the number of three-digit integers that contain no even digits is $5 \times 5 \times 5 = 125$. The requested number is therefore $999 - (9 + 20 + 25 + 100 + 125) = 720$.

JUNIOR DIVISION CONTEST NUMBER TWO FALL 2012
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F12J7 $\mathbf{20}$. Notice that $BC = AD = 3.12 + 6.88 = 10$. Thus the area of rectangle $ABCD$ is 40. In triangle BCE, the altitude from E is 4, so the area of triangle BCE is 20. Thus, the sum of the areas of triangles ABE and CDE is $40 - 20 = 20$.

F12J8 $\mathbf{5120.}$ For each committee of n people, where $1 \le n \le 10$, there are n possible chairs, so the requested number is:

$$\sum_{n=1}^{10} n\binom{10}{n} = 1\times 10 + 2\times 45 + 3\times 120 + 4\times 210 + 5\times 252 + 6\times 210 + 7\times 120 + 8\times 45 + 9\times 10 + 10\times 1$$
$$= 5120$$

Alternate solution: Each of the 10 Board members is a potential chairperson, and he or she can have any subset of the other 9 Board members as committee members. Thus the requested number is $10 \times 2^9 = 5120$.

F12J9 $\mathbf{379.}$ The requested number must be one less than a multiple of 19 and one less than a multiple of 20. Therefore, it must be one less than a multiple of $19 \times 20 = 380$. The smallest such positive integer is 379.

F12J10 $\mathbf{\frac{17}{106}}$**.** Notice that f is an odd function. That is, $f(-n) = -f(n)$ for all n. Thus,

$$\sum_{n=-5}^{6} f(n) = \sum_{n=-5}^{5} f(n) + f(6) = f(6) = \frac{17}{106}$$

F12J11 $\mathbf{20.}$ The given equation is equivalent to $(2\times 5)^5 \times n^2 = m^3$, that is, $2^5 \times 5^{10} \times n^2 = m^3$. An integer is a square if and only if each of the exponents in its prime factorization are divisible by 2, and an integer is a cube if and only if each of the exponents in its prime factorization are divisible by 3. Thus the minimum value of n^2 is $2^4 \times 5^2$, and so the requested value of n is $2^2 \times 5 = 20$.

F12J12 $\mathbf{\frac{256}{17}}$**.** Triangles ABE and ECF are similar. Thus, $\frac{4}{3} = \frac{AB}{CE} = \frac{BE + EC}{CE} = \frac{BE}{CE} + 1$. Then, $\frac{BE}{CE} = \frac{1}{3}$, so $\frac{BE}{AB} = \frac{BE}{BC} = \frac{1}{4}$. Let $BE = x$ and $AB = 4x$, and use the Pythagorean Theorem to conclude that $|ABCD| = AB^2 = 16x^2 = 16\left(\frac{16}{17}\right) = \frac{256}{17}$.

JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 2012**

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F12J13 $\mathbf{60.}$ Let x and y represent the dimensions of the rectangle, and K its area. Then $x + y = 49$, and $K = xy$. For pairs of numbers with a given sum, the closer the numbers, the greater their product. Thus K is maximum when $\{x, y\} = \{24,25\}$, so the maximum area is $24 \times 25 = 60$.

F12J14 $\mathbf{\sqrt{67}.}$ Because the inscribed angles ABD and BDC are congruent, so are $\overline{AD}$ and $\overline{BC}$. Thus $AD = BC$, and trapezoid $ABCD$ is isosceles, and so $AC = BD = 11$. Let $AD = BC = x$. Use Ptolemy's Theorem to conclude that $AB \cdot CD + AC \cdot BD = AC \cdot BD$, and then $6 \cdot 9 + x^2 = 11 \cdot 11$. Thus $AD = x = \sqrt{67}$.

F12J15 $\mathbf{2\sqrt{6}.}$ Notice that $m\angle C = 180 - (60 + 75) = 45$. Draw altitude $\overline{BD}$. Triangle BDC is $45 - 45 - 90$, so $BD = 3\sqrt{2} = \sqrt{18}$. Triangle BDA is $30 - 60 - 90$, so $DA = \frac{BD}{\sqrt{3}} = \frac{\sqrt{18}}{\sqrt{3}} = \sqrt{6}$. Hence, $AB = 2\sqrt{6}$.

F12J16 $\mathbf{2750.}$ When the product $(3 + 5 + 8 + 13 + 21)(4 + 6 + 9 + 14 + 22)$ is expanded, the result equals the requested sum. That sum is therefore $50 \cdot 55 = 2750$.

F12J17 $\mathbf{13.}$ Because $6^6 = 2^6 \times 3^6$, each divisor of 6^6 is of the form $2^a \times 3^b$, where a and b are integers between 0 and 6 inclusive. Of these, the ones that are not divisible by 6 must have $a = 0$ or $b = 0$. Thus there are $7 + 7 - 1 = 13$ of the requested divisors.

F12J18 $\mathbf{20101.}$ Let $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$. Then $4 = P(1) = a_0 + a_1 + \cdots + a_n$. So each of the a_i's is ≤ 4. Furthermore, $1276 = P(5) = a_0 5^n + a_1 5^{n-1} + \cdots + a_n$. Thus the a_i's are the digits of the base 5 representation of 1276, which is unique. Express 1276 as a base 10 numeral: $1276 = 2 \times 5^4 + 1 \times 5^2 + 1 = 20101_5$. Therefore $(x) = 2^4 + x^2 + 1$, and $P(10) = 20101$.