New York City Interscholastic Mathematics League

Senior B Division   Contest Number 1

Part I   ***Spring 2012***   ***Contest 1***   ***Time: 10 Minutes***

S12B1   Compute $123 \cdot 77 + 29 \cdot 17$.

S12B2   Compute the number of ordered triples (x, y, z) of positive <u>even</u> integers that solve the equation $x + y + z = 24$.

Part II   ***Spring 2012***   ***Contest 1***   ***Time: 10 Minutes***

S12B3   Point P is the interior of pentagon $ABCDE$ so that ΔBCP and ΔEAP are equilateral, and $CDEP$ is a square. Compute $m\angle ABP$.

S12B4   Compute all values of x that satisfy $(x^2 - 10x + 20)^2 = 3x^2 - 30x + 64$.

Part III   ***Spring 2012***   ***Contest 1***   ***Time: 10 Minutes***

S12B5   Compute the remainder when the sum $2! + 5! + 8! + \cdots + (3n-1)! + \cdots + 2012!$ is divided by 100.

S12B6   In ΔABC, $m\angle A = 60°$ and $m\angle B = 45°$. A point X is chosen randomly from the interior points of ΔABC. Compute the probability that $AX > BX$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division — CONTEST NUMBER 2

PART I — ***SPRING 2012*** — ***CONTEST 2*** — ***TIME: 10 MINUTES***

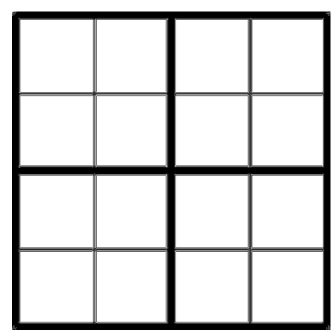
S12B7 All six faces of a cube of edge length 4 are painted red. The cube is then cut into 64 unit cubes. Finally, from each unit cube that has no paint on it, a single face is selected and painted red. Compute the total area that is painted.

S12B8 The common ratio of an infinite geometric sequence is a positive number less than 1. If the sum of all of the terms is $\frac{21}{4}$ and the sum of the first two terms is $\frac{18}{7}$, compute the common ratio.

PART II — ***SPRING 2012*** — ***CONTEST 2*** — ***TIME: 10 MINUTES***

S12B9 Squares $ABCD$ and $PQRS$ have the same center O, and the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ are parallel to $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, and $\overline{SP}$ respectively. If M is the midpoint of $\overline{DA}$ and P lies on $\overline{BM}$, compute the ratio $\frac{AB}{PQ}$.

S12B10 In the diagram to the right, a 4×4 square is partitioned into four 2×2 squares, and each 2×2 square is partitioned into four 1×1 squares. One 1×1 square is chosen from each of the four bolded 2×2 squares randomly. Compute the probability that no two of the four chosen 1×1 squares occupy the same row or column of the 4×4 square.



PART III — ***SPRING 2012*** — ***CONTEST 2*** — ***TIME: 10 MINUTES***

S12B11 Let m be an integer greater than 2. Compute the least value of m for which

$$\sum_{n=2}^{m} \log_2 n > 10$$

S12B12 The graph of the equation $2x^2 - 9xy + 10y^2 + 2x - 5y = 0$ contains a line that passes through the origin. Compute the slope of this line.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 3

PART I   ***SPRING 2012***   ***CONTEST 3***   ***TIME: 10 MINUTES***

S12B13   If $y = 1 - \sqrt{3}$, compute the numerical value of $\left(\sqrt{1 + (y-2)(y-4)} - 8\right)$.

S12B14   Given the points $A(0, 0)$, $B(5, 10)$, and $C(7, 9)$, compute the radius of the circumcircle of ΔABC.

PART II   ***SPRING 2012***   ***CONTEST 3***   ***TIME: 10 MINUTES***

S12B15   Ariel, Barry, Carly, Darryl, and Earl are sharing a bottle of juice. Ariel takes the bottle first, and as she pours juice into her glass, she spills one ounce. When her glass and the bottle have the same amount of juice, she passes the bottle to the next person. Similarly, Barry, Carly, Darryl, and Earl each spill one ounce of juice as they pour their own servings, and each stops pouring when their glass and the bottle have the same amount of juice. If each person pours his or her own glass of juice in turn and there is one ounce of juice remaining in the bottle after each person has had a turn, how many ounces of juice were in the bottle originally?

S12B16   The image of square $ABCD$ after a rotation of $45°$ about A is square $AB'C'D'$, where B' is in the interior of $ABCD$. If $AB = 1$, compute the area of the intersection of $ABCD$ and $AB'C'D'$.

PART III   ***SPRING 2012***   ***CONTEST 3***   ***TIME: 10 MINUTES***

S12B17   If θ is a positive, acute angle such that $\log(\sin\theta) - \log(\cos\theta) = \frac{1}{4}\log 4$, compute $\cos\theta$.

S12B18   Jenny is traveling next week between Sunday and Saturday and hopes that the weather will be good. On any day of this week, there is a 50% chance of rain. Given that it does not rain on Sunday or on Saturday, compute the probability, as a fraction, that there are no two or more consecutive rainy days this week.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 4

PART I ***SPRING 2012*** ***CONTEST 4*** ***TIME: 10 MINUTES***

S12B19 Compute all real values of x for which $\left||x|-4\right|=2$.

S12B20 In ΔABC, D is on $\overline{BC}$ so that $\overline{AD}$ bisects $\angle BAC$. If $AB=6$, $BD=4$, and $m\angle BDA=2\cdot m\angle BCA$, compute AD.

PART II ***SPRING 2012*** ***CONTEST 4*** ***TIME: 10 MINUTES***

S12B21 A license plate must be stamped with 3 letters in order, from a selection of 21 letters. At least one of the three letters stamped must be A, and repetition of letters is allowed. Compute the number of ways these three letters can be chosen.

S12B22 Compute $\sqrt{1+\frac{2\sqrt{3}}{2-\sqrt{3}}}$.

PART III ***SPRING 2012*** ***CONTEST 4*** ***TIME: 10 MINUTES***

S12B23 Compute the least integer n for which $\log\left(\frac{1}{2}log\left(\frac{1}{3}log\,n\right)\right)$ is a real number. (Note that log represents the base 10 logarithm.)

S12B24 Each of three jars contains a positive number of blue marbles and a positive number of red marbles. For each jar, when the proportion of blue marbles to red marbles in that jar is calculated, the sum of these three proportions is S. One marble from each jar is chosen randomly, with each marble being equally likely to be chosen. The probability that all three marbles are blue is $\frac{3}{50}$, the probability that exactly two marbles are blue is $\frac{17}{50}$, and the probability that exactly one marble is blue is $\frac{22}{50}$. Compute S.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 5

PART I ***SPRING 2012*** ***CONTEST 5*** ***TIME: 10 MINUTES***

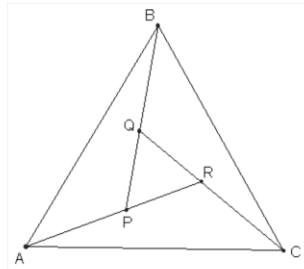
S12B25 Several circles are drawn in the plane. Exactly 19 points lie on at least two of these circles. Compute the minimum number of distinct circles drawn.

S12B26 Let $m \oplus n$ denote the number of positive integer divisors of the product mn, where m and n are positive integers. Compute $(200 \oplus 20) \oplus (350 \oplus 36)$.

PART II ***SPRING 2012*** ***CONTEST 5*** ***TIME: 10 MINUTES***

S12B27 Compute all values of x that solve the equation $3^x(3^{x+1} - 4) + 1 = 0$.

S12B28 In the accompanying diagram, points P, Q, and R are in the interior of ΔABC so that P lies on $\overline{AR}$, Q lies on $\overline{BP}$, and R lies on $\overline{CQ}$. If $AP = BQ = CR = 4$ and $PQ = QR = RP = 3$, compute the area of ΔABC.



PART III ***SPRING 2012*** ***CONTEST 5*** ***TIME: 10 MINUTES***

S12B29 Let $S_n = i^3 + i^6 + i^9 + \cdots + i^{3n}$, where $i = \sqrt{-1}$. Compute the least n greater than 2012 for which $S_n = -1$.

S12B30 Let θ be a positive acute angle for which $\csc\theta + \cot\theta = \frac{4}{3}$. Compute the numerical value of $\sin\theta$.

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NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 1

Spring 2012 Solutions

S12B1 **9964**. Each factor in the products can be written in a nicer form that invokes the difference of two squares factorization.

$$\begin{aligned}123 \cdot 77 + 29 \cdot 17 &= (100+23)(100-23) + (23+6)(23-6)\\ &= 100^2 - 23^2 + 23^2 - 6^2\\ &= 10000 - 36\\ 123 \cdot 77 + 29 \cdot 17 &= 9964\end{aligned}$$

S12B2 **55**. Any triple of positive integers that solves $X + Y + Z = 12$ corresponds to a triple of positive even integers that solves $x + y + z = 24$; multiplying X, Y, and Z by 2 will give the corresponding x, y, and z. Using the "stars and bars" method, there are $\binom{11}{2} = 55$ different triples (X, Y, Z) of positive integers that solve $X + Y + Z = 12$.

S12B3 **15° or $\frac{\pi}{12}$**. The measures of the four angles around P sum to 360°, so $m\angle APB = 150°$. ΔABP is isosceles, so $m\angle ABP = \frac{1}{2}(180° - 150°) = 15°$.

S12B4 $\{\mathbf{2, 3, 7, 8}\}$. Set $t = x^2 - 10x + 20$. Then, $t^2 = 3t + 4$, so $t = -1, 4$. Solve $x^2 - 10x + 20 = -1$, and obtain $x = 3, 7$. Solve $x^2 - 10x + 20 = 4$, and obtain $x = 2, 8$.

S12B5 **42**. When $n \geq 10$, the rightmost two digits of $n!$ are 00. So those addends do not affect the remainder. Add together $2! + 5! + 8! = 2 + 120 + 40320 = 40442$. The remainder is 42. Alternatively, calculate $8! \pmod{100}$. $8! = 8 * 7 * 6! = 56 * 6! = 50 * 6! + 6 * 6! = 50 * 6! + 6(700 + 20)$, so $8! \cong 6 * 20 \cong 20 \pmod{100}$. $2! + 5! + 8! \cong 2 + 120 + 20 \cong 42 \pmod{100}$.

S12B6 $\mathbf{\frac{3+\sqrt{3}}{12}}$ **or** $\mathbf{\frac{1}{4} + \frac{\sqrt{3}}{12}}$. The probability can be calculated by dividing the area of the region of points closer to B by the area of ΔABC. One of the boundaries of this region is the locus of points equidistant from A and B, which is the perpendicular bisector of $\overline{AB}$. This perpendicular bisector intersects $\overline{BC}$, since C is closer to A than to B. Since $m\angle B = 45°$, the region of points closer to B is an isosceles right triangle. If a, b, and c denote the side lengths of ΔABC, then the area of the isosceles right triangle is $\frac{1}{2}\left(\frac{c}{2}\right)\left(\frac{c}{2}\right) = \frac{c^2}{8}$. To find the area of ΔABC, let D be a point on $\overline{AB}$ so that $\overline{CD}$ is an altitude, and let $h = CD$. Then, $AD = \frac{h}{\tan 60°} = \frac{h\sqrt{3}}{3}$ and $BD = \frac{h}{\tan 45°} = h$. So, $c = \frac{h\sqrt{3}}{3} + h$, or $h = \frac{c}{1+\frac{\sqrt{3}}{3}}$. Then, the area of ΔABC is $\frac{1}{2}\frac{c^2}{1+\frac{\sqrt{3}}{3}}$. The probability that $AX > BX$ is $\frac{c^2}{8} \div \left(\frac{1}{2}\frac{c^2}{1+\frac{\sqrt{3}}{3}}\right) = \frac{3+\sqrt{3}}{12}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 2

Spring 2012 Solutions

S12B7 **104**. The surface area of the original cube is $6 \cdot 4^2 = 96$. After the original cube is cut into unit cubes, only eight unit cubes in the center are completely unpainted. So, eight more faces are painted and in total, the painted surface area is $96 + 8 = 104$.

S12B8 $\mathbf{\frac{5}{7}}$. Let a be the first term and let r be the common ratio. Then, $\frac{21}{4} = \frac{a}{1-r}$ and $\frac{18}{7} = a + ar = a(1+r)$. Solve the second equation for a and substitute into the first: $\frac{21}{4} = \frac{\frac{18}{7}}{(1+r)(1-r)}$. From here, isolate r: $r = \frac{5}{7}$.

S12B9 **3**. Let T be the intersection of OM and SP. Without loss of generality, set $PQ = 1$ to simplify the algebra. $OT = PT = \frac{1}{2}$. Since ΔABM is similar to ΔTMP, $TM = 2(PT) = 1$. Then, $OM = OT + TM = \frac{3}{2}$, so $AB = 3$. Then $\frac{AB}{PQ} = 3$.

S12B10 $\mathbf{\frac{1}{16}}$. Count the number of ways to choose the 1×1 squares one at a time. Start at the top left 2×2 square. Any square can be picked, so there are 4 choices. Moving to the 2×2 square at the top right, the first square chosen occupies the same row as two of the 1×1 squares, so there are only 2 choices. Similarly, there are 2 choices in the bottom left square. Once three squares are chosen, there is only one unoccupied row and one unoccupied column, so there is 1 choice for the last square. The correct probability is $\frac{4\cdot 2\cdot 2\cdot 1}{4\cdot 4\cdot 4\cdot 4} = \frac{1}{16}$

S12B11 **7**. Using the rules for logarithms,

$$\sum_{n=2}^{m} \log_2 n = \log_2 2 + \log_2 3 + \cdots + \log_2 m$$
$$= \log_2(2 \cdot 3 \cdot \ldots \cdot m)$$
$$= \log_2 m!$$

Since $10 = \log_2 1024$, we require $m! > 1024$. Since $6! = 720$ and $7! = 5040$, $m = 7$.

S12B12 $\mathbf{\frac{2}{5}}$. The equation can be factored as $(2x - 5)(x - 2y + 1) = 0$, from which $2x - 5y = 0$ or $x - 2y + 1$=0. The first gives the only line, $y = \frac{2}{5}x$, that passes through the origin.
Alternatively, suppose the line has equation $y = mx$. Substituting this in, we get that $2x^2 - 9x(mx) + 10(mx)^2 + 2x - 5mx = 0$ must hold for <u>all</u> x. Since this is a polynomial in x, it follows that the coefficients must be equal to 0. Thus, $2 - 5m = 0$ and the only possibility is $m = \frac{2}{5}$. We can then check that $y = \frac{2}{5}x$ indeed always a solution to the given equation. Plugging in, we get $2x^2 - 9x\left(\frac{2}{5}x\right) + 10\left(\frac{2}{5}x\right)^2 + 2x - 5\left(\frac{2}{5}x\right) = 2x^2 - \frac{18}{5}x^2 + \frac{8}{5}x^2 + 2x - 2x = 2x^2 - \frac{10}{5}x^2 = 0$.

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Senior B Division   CONTEST NUMBER 3

Spring 2012 Solutions

S12B13 $\mathbf{-6+\sqrt{3}}$. First, simplify the expression:

$$\sqrt{1+(y-2)(y-4)}-8=\sqrt{1+y^2-6y+8}-8$$
$$=\sqrt{y^2-6y+9}-8$$

Recall that the principal square root of a positive real number is the positive square root. Since $\mathrm{y}-3=-2-\sqrt{3}<0, \sqrt{(\mathrm{y}-3)^2}=3-\mathrm{y}$.

$$\sqrt{y^2-6y+9}-8=3-y-8$$
$$=-6+\sqrt{3}$$

S12B14 $\mathbf{\frac{\sqrt{130}}{2}}$. The circumcenter must be equidistant from all vertices. Let its coordinates be (x,y). Then, the following must be true:

$$\sqrt{x^2+y^2}=\sqrt{(x-5)^2+(y-10)^2}$$
$$\sqrt{x^2+y^2}=\sqrt{(x-7)^2+(y-9)^2}$$

Expanding the squared terms, we have:

$$\sqrt{x^2+y^2}=\sqrt{x^2-10x+25+y^2-20x+100}=\sqrt{x^2+y^2-10x-20y+125}$$
$$\sqrt{x^2+y^2}=\sqrt{x^2-14x+49+y^2-18y+81}=\sqrt{x^2+y^2-14x-18y+130}$$

For this to be true, we must have:

$$130-14x-18y=0$$
$$125-10x-20y=0$$

Solving these simultaneously, we get $(x,y)=\left(\frac{7}{2},\frac{9}{2}\right)$. Taking the distance from $(0,0)$, the circumradius is $\sqrt{\left(\frac{7}{2}\right)^2+\left(\frac{9}{2}\right)^2}=\frac{\sqrt{130}}{2}$.

S12B15   **63**. Working backwards, when Earl took the bottle from Darryl, the bottle contained the leftover 1 ounce, the 1 ounce in Earl's glass, and the 1 ounce that Earl spills, or 3 ounces total. When Darryl took the bottle from Carly, the bottle contained the 3 ounces Darryl gave to Earl, the 3 ounces that Darryl took for himself, and the 1 that Darryl spills, or 7 ounces total. Continuing this pattern of doubling and adding 1 ounce, the bottle would have contained 1, 3, 7, 15, 31, and 63 ounces after 5, 4, 3, 2, 1, and 0 people poured from it, respectively.

S12B16 $\mathbf{\sqrt{2}-1}$. Let the intersection of $\overline{B'C'}$ and $\overline{CD}$ be P. The intersection of the two squares, $AB'PD$, is a kite that can be bisected into two congruent right triangles, $\Delta AB'P$ and ΔADP. The area of ΔAPB can be found if $B'P$ is known. Since C is collinear with A and B', $B'C=AC-AB'=\sqrt{2}-1$. Since $CB'P$ is an isosceles right triangle, $B'P=B'C=\sqrt{2}-1$. So, the area of the right triangle $AB'P$ is $\frac{1}{2}(AB')(B'P)=\frac{\sqrt{2}-1}{2}$, and therefore, area of $AB'PD$ is $2\cdot\frac{\sqrt{2}-1}{2}=\sqrt{2}-1$.

S12B17 $\mathbf{\frac{\sqrt{3}}{3}}$. Using the rules of logarithms, obtain $\frac{\sin\theta}{\cos\theta}=\sqrt[4]{4}$, or simply, $\tan\theta=\sqrt{2}$. By the Pythagorean identity, $\sec^2\theta=1+\tan^2\theta=3$. Then, $\sec\theta=\sqrt{3}$ and $\cos\theta=\frac{\sqrt{3}}{3}$. Note that the negative value is rejected because θ is an acute angle.

S12B18 $\mathbf{\frac{13}{32}}$. If there are 0 or 1 rainy days in the week, there cannot be 2 consecutive rainy days. There is 1 way for exactly 0 days to be rainy and 5 ways for exactly 1 days to be rainy. If there are 2 rainy days, we can order 2 rainy and 2 sunny days without restriction, then insert a sunny day somewhere between the 2 rainy days. Then, there are $\binom{4}{2} = 6$ ways to have 2 rainy days ordered so that there are no consecutive days that are rainy. Finally, there is 1 way to order 3 rainy days so that no consecutive days are rainy, and 0 ways if there are 4 or 5 rainy days. Altogether, there are $1 + 5 + 6 + 1 + 0 + 0 = 13$ ways to have a week without consecutive rainy days and $2^5 = 32$ different possibilities, so the desired probability is $\frac{13}{32}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 4

Spring 2012 Solutions

S12B19 $\mathbf{\pm 2, \pm 6}$. Consider the two cases: $|x| - 4 = 2$ and $|x| - 4 = -2$. From the first case, $|x| = 6$, so $x = \pm 6$. From the second case, $|x| = 2$, so $x = \pm 2$.

S12B20 $\mathbf{5}$. Since $\angle BDA$ is an exterior angle of ΔADC, its measure is the sum of the measures of the remote interior angles. Then, $m\angle CAD = m\angle ACD$. Also, $\overline{AD}$ bisects $\angle BAC$, so $m\angle DAB = m\angle CAD = m\angle ACD$. Then, ΔABC is similar to ΔDBA, by angle-angle similarity. Solving the proportion $\frac{AB}{BC} = \frac{DB}{BA}$. obtain $BC = 9$, which implies $CD = 5$. Since ΔDAC is isosceles with congruent base angles $\angle CAD$ and $\angle ACD$, we also have $AD = 5$.

S12B21 $\mathbf{1261}$. Using complementary counting, first ignore the restriction that A must be used. Since repetition is allowed, there are 21 choices for each of the three letters, so there are $21^3 = 9261$ ways to choose the three letters. Of these, there are $20^3 = 8000$ permutations that use only the 20 letters besides A. Then there are $9261 - 8000 = 1261$ ways to choose the three letters while obeying all the restrictions.

S12B22 $\mathbf{2 + \sqrt{3}}$. Add fractions before rationalizing the denominator:

$$\begin{aligned}\sqrt{1 + \frac{2\sqrt{3}}{2 - \sqrt{3}}} &= \sqrt{\frac{2 - \sqrt{3}}{2 - \sqrt{3}} + \frac{2\sqrt{3}}{2 - \sqrt{3}}} \\ &= \sqrt{\frac{2 + \sqrt{3}}{2 - \sqrt{3}}} \\ &= \sqrt{\frac{2 + \sqrt{3}}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}}} \\ &= \sqrt{\frac{\left(2 + \sqrt{3}\right)^2}{1}} \\ &= 2 + \sqrt{3}\end{aligned}$$

S12B23 $\mathbf{1001}$. For $\log x$ to be real, x must be positive. Then $\frac{1}{2}\log\left(\frac{1}{3} log\, n\right) > 0$. For $\log x$ to be positive, x must be greater than 1. Then, $\frac{1}{3}\log n > 1$. Therefore, $n > 10^3 = 1000$. The least integer value of n is $1000 + 1 = 1001$.

S12B24 $\mathbf{\frac{11}{4}}$. Let p_1, p_2, and p_3 represent the probabitlities that a blue marble is drawn from each jar, and let q_1, q_2, and q_3. Represent the corresponding probabilities for the red marbles. Note that $S = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \frac{p_3}{q_3}$. Each of the given probabilities can be expressed as follows:

$$p_1 p_2 p_3 = \frac{3}{50} \qquad (1)$$

$$p_1 p_2 q_3 + p_1 q_2 p_3 + q_1 p_2 p_3 = \frac{17}{50} \qquad (2)$$

$$p_1 q_2 q_3 + q_1 p_2 q_3 + q_1 q_2 p_3 = \frac{22}{50} \qquad (3)$$

The probability that no blue marbles are drawn is $1-\left(\frac{3}{50}+\frac{17}{50}+\frac{22}{50}\right)=\frac{8}{50}$. Then, $q_1q_2q_3=\frac{8}{50}$. Divide equation (3) by this expression to obtain $\frac{p_1}{q_1}+\frac{p_2}{q_2}+\frac{p_3}{q_3}=\frac{22}{8}=\frac{11}{4}$. (This system can be used to solve for the quantities $\frac{p_1}{q_1}, \frac{p_2}{q_2}$, and $\frac{p_3}{q_3}$. Coupled with $p_i+p_i=1$, we can find the p_i's to be $\frac{1}{2}, \frac{1}{5}$, and $\frac{3}{5}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 5

Spring 2012 Solutions

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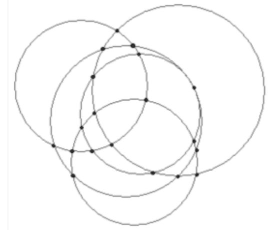

S12B25 **5**. Let n be the number of circles drawn. Any pair of distinct circles can intersect in at most two points. Then, there are at most $2 \cdot \binom{n}{2} = n^2 - n$ intersection points. If $n^2 - n \geq 19$, then $n \geq 5$. Here are 5 circles positioned so that there are 19 intersections points.

S12B26 **28**. The product $200 \cdot 20 = 2^5 \cdot 5^3$ has $(5+1)(3+1) = 24$ positive integer divisors, so $200 \oplus 20 = 24$. The product $350 \cdot 36 = 2^3 \cdot 3^2 \cdot 5^2 \cdot 7^1$ has $(3+1)(2+1)(2+1)(1+1) = 72$ positive integer divisors, so $350 \oplus 36 = 72$. Then, the product $24 \cdot 72 = 2^6 \cdot 3^3$ has $(6+1)(3+1) = 28$ positive integer divisors. Therefore, $24 \oplus 72 = 28$.

S12B27 $\{\mathbf{0}, \mathbf{-1}\}$. Substitute $y = 3^x$ and solve $y(3 \cdot y - 4) + 1 = 0$, or $3y^2 - 4y + 1 = (3y-1)(y-1) = 0$, to obtain $y = 1$ or $y = \frac{1}{3}$. Then $x = 0$ or $x = -1$.

S12B28 $\mathbf{\frac{93\sqrt{3}}{4}}$. Since ΔPQR is equilateral, its exterior angles are all $120°$. Then by SAS, triangles ΔAPB, ΔBQC, and ΔCRA are congruent, and $AB = BC = CA$. To find the side length of equilateral triangle ΔABC, apply the Law of Cosines to ΔAPB:

$$AB^2 = AP^2 + PB^2 - 2(AP)(PB)\cos\angle APB = 4^2 + 7^2 - 2(4)(7)\left(-\frac{1}{2}\right) = 93$$

Then the area of ΔABC is $\frac{AB^2\sqrt{3}}{4} = \frac{93\sqrt{3}}{4}$.

S12B29 **2015**. Observe that the sequence $\{S_n\} = \{-i, -i-1, -1, 0, -i, -i-1, -1, 0, \ldots\}$ is periodic. Then, $S_n = -1$ when n is 1 less than a multiple of 4. The least integer greater than 2012 with this property is 2015.

S12B30 $\mathbf{\frac{24}{25}}$. Using the identity $\csc^2\theta - \cot^2\theta = 1$, obtain $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$, so $\csc\theta - \cot\theta = \frac{3}{4}$. Solving simultaneously with $\csc\theta + \cot\theta = \frac{4}{3}$ for $\csc\theta$ gives $2\csc\theta = \frac{4}{3} + \frac{3}{4}$, or $\csc\theta = \frac{25}{24}$. Then, $\sin\theta = \frac{24}{25}$.