New York City Interscholastic Math League

Soph-Frosh Division   Contest Number 1

Part I   FALL 2011   Contest 1   Time: 10 Minutes

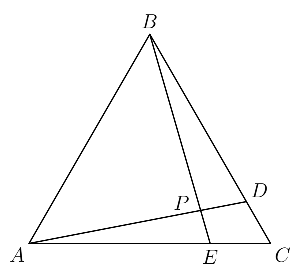
F11SF1 The length and width of a rectangle are consecutive integers. The diagonal of the rectangle is 5. Compute the area of the rectangle.

F11SF2 Suppose that the number x satisfies the equation $x+x^{-1}=3$. Compute the value of x^8+x^{-8}.

Part II   FALL 2011   Contest 1   Time: 10 Minutes

F11SF3 A standard six-sided dice is rolled, and S is the sum of the five numbers that are visible. Compute the probability that S is a prime number.

F11SF4 In $\triangle ABC$, points D and E lie on BC and AC, respectively. If AD and BE intersect at P so that $\frac{AP}{DP}=5$ and $\frac{BP}{EP}=7$, compute the ratio, CD/BD.



Part III   FALL 2011   Contest 1   Time: 10 Minutes

F11SF5 A parabola $y=x^2$ and a line $x=\frac{y}{4}-3$ intersect at two points $A(a,b)$ and $B(c,d)$. Compute the sum $a+b+c+d$.

F11SF6 The polynomial $x^3-ax^2+bx-455$ has three positive prime roots. If one of the roots is 5, then compute $a+b$.

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Soph-Frosh Division — Contest Number 2

Part I | FALL 2011 | Contest 2 | Time: 10 Minutes

F11SF7 Suppose $4x + 3y = 6x + y \neq 0$. Compute $\frac{6x-2y}{x+y}$.

F11SF8 Two positive integers a and b satisfy the following equation:

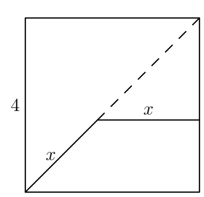
$$(a + 2b)(a - b) = 10$$

Find all possible values for $2a + 3b$.

Part II | FALL 2011 | Contest 2 | Time: 10 Minutes

F11SF9 A three-digit number is formed using the digits 1, 2, 3, 5, 7, and 8 (repetition of the digits is not allowed). Compute the probability that the three-digit number is less than 500.

F11SF10 A square with a side length of 4 is shown in the diagram below. Take the point on the dotted diagonal that is equally distant from the bottom left corner and the right side. This distance can be expressed as $a - b\sqrt{2}$. Compute $a + b$ where a and b are both integers.



Part III | FALL 2011 | Contest 2 | Time: 10 Minutes

F11SF11 Positive integers x and y satisfy the following equation: $\sqrt{xy} = 6$. Compute the number of distinct pairs (x,y) that satisfy the above equation.

F11SF12 Trapezoid $ABCD$ has $AD||BC$, $BD = 1$, $\angle DBA = 33°$, and $\angle BDC = 66°$. Given that the ratio $BC : AD$ is 7:3, compute CD.

New York City Interscholastic Math League
Soph-Frosh Division   Contest Number 3

Part I FALL 2011 Contest 3 Time: 10 Minutes

F11SF13 A function is defined as

$$f(\sqrt{5t-1}) = \frac{t+3}{3t-4} \text{ where } t > 0.2 \text{ and } t \neq \frac{4}{3}$$

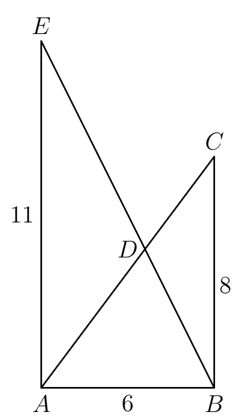
Compute the value of $f(3)$.

F11SF14 If $x^2 + y^2 = 13$ and $x^2 - y^2 = 3$, compute $|xy|$.

Part II FALL 2011 Contest 3 Time: 10 Minutes

F11SF15 From a group of boys and girls, 6 girls leave. Then there are two boys for each girl. After 10 boys leave the remaining group, there are 3 girls for each boy. Compute the number of boys in the beginning.

F11SF16 In the figure, $\angle EAB$ and $\angle ABC$ are right angles. Given that $AB = 6$, $BC = 8$, and $AE = 11$, compute the difference between the areas of $\triangle ADE$ and $\triangle BDC$



F11SF17 Compute the number of positive integers x less than or equal to 16 that satisfy the following condition:

$$x! \text{ is divisible by } 1 + 2 + 3 + \cdots + x$$

F11SF18 When 30 is appended to a list of integers, the mean is increased by 4. When 2 is appended to the enlarged list, the mean of the enlarged list is decreased by 2. Compute the sum of the elements in the original list.

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Soph-Frosh Division — Contest Number 1 Solutions

F11SF1. **12** One of the famous Pythagorean triples is (3,4,5). In other words, $3^2+4^2=5^2$. Hence, we know that the length and width of the rectangle are 3 and 4. The area of the rectangle is $3\times 4=12$. This can be also solved by solving a quadratic equation. Let l and w be the length and width of the rectangle, respectively. Without loss of generality, assume that $l<w$. Since l and w are consecutive integers, we have $l+1=w$. Using the Pythagorean Theorem, we have:

$$l^2+w^2=l^2+(l+1)^2=2l^2+2l+1=25$$

$$l^2+l-12=0\Longrightarrow (l-3)(l+4)=0$$

Since l is a positive integer, we have $l=3$ and $w=l+1=4$. The area of the rectangle is $l\times w=12$.

F11SF2. **2207** We are given that

$$x+\frac{1}{x}=3$$

Squaring both sides of the above equation results in

$$x^2+\frac{1}{x^2}+2=9\Longrightarrow x^2+\frac{1}{x^2}=7$$

We can repeat the process to get:

$$x^4+\frac{1}{x^4}=47$$

$$x^8+\frac{1}{x^8}=2207$$

F11SF3. **1/3** The sum of all six numbers on a dice is:

$$1+2+\cdots+6=\frac{7\cdot 6}{2}=21$$

If one is not visible, then $S=21-1=20$. Hence, there are six possible values for S: 20, 19, 18, 17, 16, and 15. Among these six numbers, only 17 and 19 are prime numbers. Therefore, the probability that S is a prime number is $\frac{2}{6}=\frac{1}{3}$.

F11SF4. **3/17** Let the area of BPD, denoted as $[BPD]$, be x. Then from the ratio $AP/DP=5$, we know that $[BPA]=5x$. Using the ratio $BP/EP=7$, then the area of APE is

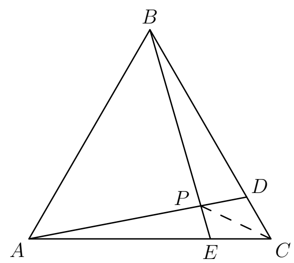
$$\frac{5x}{[APE]}=7\Longrightarrow [APE]=\frac{5x}{7}$$

Now let $[PDC] = y$. From the ratio $BP/EP = 7$, we have $[PEC] = (x+y)/7$.

$$\frac{[PDC]}{[APC]} = \frac{1}{5} \Longrightarrow \frac{y}{\frac{5}{7}x + \frac{x+y}{7}} = \frac{1}{5}$$

$$\frac{y}{\frac{6x+y}{7}} = \frac{7y}{6x+y} = \frac{1}{5} \Longrightarrow 35y = 6x + y \Longrightarrow y = \frac{3}{17}x$$

$$\frac{CD}{BD} = \frac{[PDC]}{[BPD]} = \frac{y}{x} = \frac{3}{17}$$



F11SF5. **44** Solving for y in the line equation gives us:

$$y = 4x + 12$$

Since the parabola and the line intersect, we can set the two equations equal to each other:

$$x^2 = 4x + 12 \Longrightarrow x^2 - 4x - 12 = 0 \Longrightarrow (x-6)(x+2) = 0$$

Solving this quadratic equation results in $x = 6$ and $x = -2$. If $x = 6$, then $y = x^2 = 36$. If $x = -2$, then $y = x^2 = 4$. Therefore, the two points are $A(6, 36)$ and $B(-2, 4)$. The answer is $6 + 36 - 2 + 4 = 44$.

F11SF6. **216** Let x_1, x_2, and x_3 be the roots of the polynomial. Using Vieta's Formula, we have that

$$x_1 + x_2 + x_3 = -\frac{(-a)}{1} = a \quad (1)$$

$$x_1x_2 + x_1x_3 + x_2x_3 = b \quad (2)$$

$$x_1x_2x_3 = -\frac{(-455)}{1} = 455$$

Since one of the zeros is 5, without loss of generality, let $x_1 = 5$:

$$x_2x_3 = 91$$

91 has only two prime factors, namely 7 and 13. We are given that all three zeros are prime numbers, and WLOG set $x_2 = 7$ and $x_3 = 13$. From eq. (1) and (2), we can compute a and b:

$$a = x_1 + x_2 + x_3 = 5 + 7 + 13 = 25$$

$$b = x_1x_2 + x_1x_3 + x_2x_3 = 35 + 65 + 91 = 191$$

Therefore, $a + b = 216$.

F11SF7. **2** Rearranging the given equation gives us:

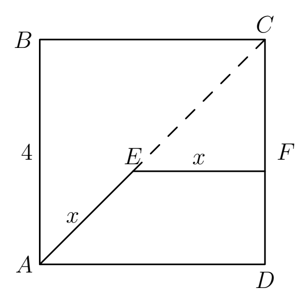
$$2x = 2y \Longrightarrow x = y$$

$$\text{Then, } \frac{6x-2y}{x+y} = \frac{6x-2x}{x+x} = 2$$

F11SF8. **9, 17** Since a and b are positive integers, it is clear that $a+2b$ and $a-b$ are integers and $a+2b > a-b$. Thus we can either have $a+2b=5$ and $a-b=2$ or $a+2b=10$ and $a-b=1$. Solving the first set of equations, we get $a=3$ and $b=1$. Therefore, $2a+3b = 2(3)+3(1) = 9$. Solving the second set of equations, we get $a=4$ and $b=3$. Therefore, $2a+3b = 2(4)+3(3) = 17$.

F11SF9. **1/2** Since the three-digit number has to be less than 500, only 1, 2, and 3 are allowed for the hundred's place. There is no restriction for the ten's and one's places. Therefore, the probability that the three-digit number is less than 500 is $\frac{3}{6} = \frac{1}{2}$.

F11SF10. **12** First we can label the figure. $\triangle CEF$ and $\triangle CAD$ are similar:



$$\frac{EF}{AD} = \frac{x}{4} = \frac{CF}{4} \Longrightarrow CF = x$$

Using the Pythagorean Theorem, we can solve for CE:

$$CE = \sqrt{2x^2} = x\sqrt{2}$$

$$CA = 4\sqrt{2} = AE + CE = x + x\sqrt{2}$$

$$4\sqrt{2} = x(1+\sqrt{2}) \Longrightarrow x = \frac{4\sqrt{2}}{1+\sqrt{2}} = 8 - 4\sqrt{2}$$

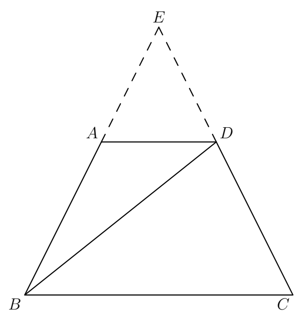
Then $a = 8$ and $b = 4$. $a + b = 12$.

F11SF11. $\boxed{\mathbf{9}}$ First, square both sides of the given equation to get:

$$xy = 36$$

Then we can list out all the possible pairs: (1,36), (2,18), (3,12), (4,9), (6,6), (9,4), (12,3), (18,2), (36,1). There are 9 pairs.

F11SF12. $\boxed{\mathbf{4/3}}$ Extend AB and CD to meet at E as shown in the diagram below (figure not drawn to scale): We have $\angle DBA = 33°$ and $\angle BDE = 180 - 66 = 114°$. Then



$\angle BED = 33°$, and $\triangle BDE$ is an isosceles triangle. Hence $DE = BD = 1$. Since $AD||BC$, $\triangle ADE$ and $\triangle BCE$ are similar.

$$\frac{AD}{BC} = \frac{3}{7} = \frac{DE}{DE + CD} = \frac{1}{1 + CD} \Longrightarrow CD = \frac{4}{3}$$

New York City Interscholastic Math League
Soph-Frosh Division — Contest Number 3 Solutions

F11SF13. $\boxed{\mathbf{5/2}}$ First, we need to solve for t in $\sqrt{5t-1}=3$.

$$\sqrt{5t-1}=3 \Longrightarrow 5t-1=9 \Longrightarrow t=2$$

$$f(3)=\frac{2+3}{6-4}=\frac{5}{2}$$

F11SF14. $\boxed{\mathbf{2}\sqrt{10}}$ Adding the two given equations gives us:

$$2x^2=16 \Longrightarrow x^2=8$$

$$y^2=13-x^2=5$$

$$|xy|=\sqrt{x^2y^2}=\sqrt{40}=2\sqrt{10}$$

F11SF15. $\boxed{\mathbf{12}}$ Let B and G be the number of boys and girls in the beginning, respectively. After 6 girls leave, we have the following equation:

$$B=2(G-6)=2G-12$$

After 10 boys leave the group, we have

$$3(B-10)=G-6$$

Substituting $B=2G-12$ into $3(B-10)=G-6$ and solving for G, we get

$$3(2G-22)=G-6 \Longrightarrow 6G-66=G-6 \Longrightarrow G=12$$

Then $B=2G-12=12$.

F11SF16. $\boxed{\mathbf{9}}$ Let the area of $\triangle ABD$, denoted as $[ABD]$, be x. Then we have

$$[ADE]+x=33$$

$$[BDC]+x=24$$

Subtracting the two equations gives us the difference between $[ADE]$ and $[BDC]$:

$$[ADE]-[BDC]=33-24=9$$

F11SF17. $\boxed{\mathbf{10}}$ Using the arithmetic sum equation, we can express $1+2+3+\cdots x$ as

$$1+2+3+\cdots x=\frac{(1+x)x}{2}$$

To satisfy the condition, we need to have

$$\frac{x!}{\frac{(1+x)x}{2}} = k$$

where k is an integer. We can simplify the equation:

$$\frac{2x!}{(1+x)x} = \frac{2(x-1)!}{x+1} = k$$

k is an integer if and only if $x+1$ is not an odd, prime number. There are 6 odd prime numbers less than or equal to $x+1 = 16+1 = 17$: 3, 5, 7, 11, 13 and 17. Therefore, there are 16 - 6 = 10 positive integers less than or equal to 16 that satisfy the condition.

F11SF18. $\boxed{\mathbf{40}}$ Let S and n denote the sum and number of the elements in the original list, respectively. When 30 is appended to the list, the mean is increased by 4:

$$\frac{S+30}{n+1} = \frac{S}{n} + 4 = \frac{S+4n}{n}$$

Simplifying the above equation gives us:

$$Sn + 30n = Sn + S + 4n^2 + 4n \Longrightarrow 4n^2 + S = 26n$$

When 2 is appended to the enlarged list, the mean is decreased by 2:

$$\frac{S+32}{n+2} = \frac{S}{n} + 2 = \frac{S+2n}{n}$$

Again, simplifying the equation results in:

$$Sn + 32n = Sn + 2S + 2n^2 + 4n \Longrightarrow n^2 + S = 14n$$

Finally subtracting $n^2 + S = 14n$ from $4n^2 + S = 26n$, we get:

$$3n^2 = 12n \Longrightarrow n(n-4) = 0$$

Since n cannot be a zero, $n = 4$. To find S, we simply substitute $n = 4$ into either $n^2 + S = 14n$ or $4n^2 + S = 26n$ to get $S = 40$.