NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 1

PART I   ***FALL 2011***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F11B1   Let a and b be integers such that $a+b=49$. Compute the maximum value of the product ab.

F11B2   Let z be a complex number that satisfies $z+6i=zi$. If a and b are the real and imaginary parts of z, respectively, compute the ordered pair (a,b).

PART II   ***FALL 2011***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F11B3   An equilateral triangle and a semicircle have the same area. Let s be the side length of that triangle, and let r be the radius length of that semicircle. Compute the ratio of the area of an equilateral triangle with side length r to the area of a semicircle with radius length s.

F11B4   Compute x if $2^{\log_4 6} = 6^{\log_8 x}$.

PART III   ***FALL 2011***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F11B5   Compute x for which the sequence $3x, 2x^2, x^3$ is an arithmetic sequence with a positive common difference.

F11B6   Compute

$$\left\lfloor \sum_{n=1}^{6} (\tan^n 60° + \cot^n 60°)^2 \right\rfloor$$

Where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 2

PART I	***FALL 2011***	***CONTEST 2***	***TIME: 10 MINUTES***

F11B7 Let n be a positive integer less than 1000. If n^3 has 10 factors, compute the largest value of n.

F11B8 Points P, Q, R, and S are chosen on the sides of parallelogram $ABCD$, so that P is on $\overline{AB}$, Q is on $\overline{BC}$, R is on $\overline{CD}$, S is on $\overline{DA}$, and $AP = BQ = CR = DS = \frac{1}{3}AB$. Compute the ratio of the area of $PQRS$ to the area of $ABCD$.

PART II	***FALL 2011***	***CONTEST 2***	***TIME: 10 MINUTES***

F11B9 The roots of the equation $3x^3 - 38x^2 + cx - 192 = 0$ form a geometric progression. Compute c.

F11B10 In ΔABC, $\angle A = 45°$, $\angle B = 60°$, and $AC = \sqrt{15}$. D is also a point on $\overline{AB}$ so that $\overline{AB} \perp \overline{CD}$. The circle with diameter $\overline{AB}$ intersects $\overline{CD}$ at point E. Compute $(DE)^2$.

PART III	***FALL 2011***	***CONTEST 2***	***TIME: 10 MINUTES***

F11B11 The number 2011 has the property that one of its digits is the sum of its other digits, i.e., $0 + 1 + 1 = 2$. Compute the sum of the two largest integers less than 2011 with this property.

F11B12 Compute the product of the *nonreal* roots of the equation $x^4 + 4x^3 + 6x^2 + 1004x + 1001 = 0$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 3

PART I ***FALL 2011*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F11B13 Al is doing poorly on his calculus tests and wants to improve his average score with a great score on his next test. If he scores 90, his average will increase by 1 point. If he scores 100, his average will increase by 2 points. Compute Al's current test average.

F11B14 Suppose that $f(x)$ is a real-valued function such that $f(x^2+x)=(x+1)\left(x^3+f(x)\right)$ holds for all real values of x. Compute $2f(-3)+3f(2)$.

PART II ***FALL 2011*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F11B15 In rectangle $ABCD$, $AB=1$ and $BC=\sqrt{2}$. Point P is not coplanar with $ABCD$, and $PA=PB=PC=PD=1$. Compute the volume of the rectangular pyramid $PABCD$.

F11B16 In ΔABC, $AB=5$, $BC=4$, and $CA=3$. Point P is the intersection, other than point C, of the circle that circumscribes ΔABC and the angle bisector of $\angle ACB$. Compute PC.

PART III ***FALL 2011*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F11B17 How many perfect squares are between 10000 and 20000, inclusive?

F11B18 A sequence is called *organized* if two of its terms have the same value only if every term between them also shares this value. For example, the sequence {4,4,4,3,1,1,2,2,2,2} is organized, while {1,1,2,2,3,3,3,2,2,2} is *not* organized. Consider the collection of sequences of ten terms, each of whose terms is a member of {1,2,3,4}. How many of these sequences are organized?

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 4

PART I ***FALL 2011*** ***CONTEST 4*** ***TIME: 10 MINUTES***

F11B19 Segments $\overline{AB}$ and $\overline{CD}$ are chords in a circle that intersect at E. If $AE = 7$, $EB = 15$, and $CD = 26$, compute the shorter of CE and DE.

F11B20 For how many integer values of n will the expression $\frac{5n-12}{n-6}$ also have an integer value?

PART II ***FALL 2011*** ***CONTEST 4*** ***TIME: 10 MINUTES***

F11B21 The lengths of the three sides of a nondegenerate triangle are x^2, $5x + 15$, and 9. If x is an integer, compute x.

F11B22 In ΔABC, $AB = 7$, $BC = 9$, and $CA = 10$. Let I be the incenter of ΔABC. If P is a point on $\overline{BC}$ so that the area of quadrilateral $ABPI$ is $\frac{4}{9}$ of the area of quadrilateral $AIPC$, compute BP.

PART III ***FALL 2011*** ***CONTEST 4*** ***TIME: 10 MINUTES***

F11B23 Let α, β, and γ be angles such that $0 \leq \alpha \leq \beta \leq \gamma$, $\alpha + \beta + \gamma = \pi$, and $\sin\alpha + \sin\beta = \sin\gamma$. Compute α.

F11B24 A bag contains 3 red marbles and 2 green marbles. Jill randomly pulls marbles out of the bag, one at a time, without replacement, until the marbles remaining inside of the bag are all the same color. Compute the probability that Jill pulls out exactly 3 marbles.

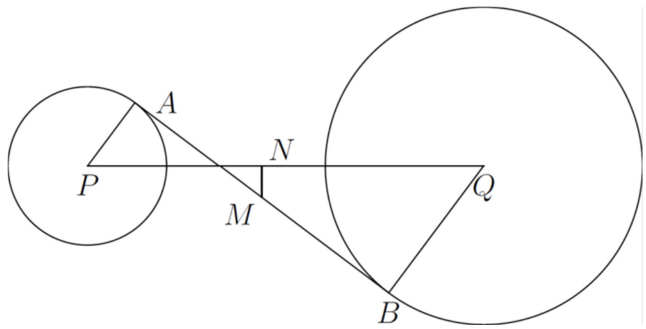
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 5

PART I ***FALL 2011*** ***CONTEST 5*** ***TIME: 10 MINUTES***

F11B25 The sum of four positive integers is 9. Compute the maximum value of their product.

F11B26 $\overline{PA}$ is the radius of a circle with center P, and $\overline{QB}$ is the radius of a circle with center Q, so that $\overline{AB}$ is a common internal tangent of the two circles. Let M be the midbout of $\overline{AB}$ and N be the point of $\overline{PQ}$ so that $\overline{MN} \perp \overline{PQ}$. If $PA = 5$, $QB = 10$, and $PQ = 17$, compute PN.



PART II ***FALL 2011*** ***CONTEST 5*** ***TIME: 10 MINUTES***

F11B27 Compute the least positive integer n for which $\left(-\sqrt{2} + i\sqrt{6}\right)^n$ will be an integer, where i is the imaginary unit.

F11B28 "When translated upward by b units, the graph of the equation $y = x^4 - 18x^2$ will have four distinct x-intercepts." Compute the minimum positive value of b for which the statement is false.

PART III ***FALL 2011*** ***CONTEST 5*** ***TIME: 10 MINUTES***

F11B29 The areas of three noncongruent circles form a geometric sequence. The radius of the largest circle is eight times the radius of the smallest. The length of the radius of the middle circle is 8. Compute the area of the smallest circle.

F11B30 For natural numbers n, let $f(n)$ be 5 times the sum of the digits of n. (If n has one digit, then $f(n) = 5n$.) The sequence $\{a_i\}_{i=1}^{\infty}$ is defined recursively by $a_1 = 2011$ and $a_{i+1} = f(a_i)$. Compute a_{2011}.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 1

Fall 2011 Solutions

F11B1 **600**. By the arithmetic mean-geometric mean inequality, $\frac{a+b}{2} \geq \sqrt{ab}$, so $ab \leq \left(\frac{a+b}{2}\right)^2 = \frac{2401}{4}$. Since a and b are positive integers (otherwise their product would be less than or equal to 0), $ab \leq \left\lfloor\frac{2401}{4}\right\rfloor = 600$. This maximum value is achieved when $a = 25$ and $b = 24$.

F11B2 $(\mathbf{3}, -\mathbf{3})$. Let $z = a + bi$. Then $z + 6i = zi$ implies $a + (b + 6)i = -b + ai$, from which we obtain $a = -b$ and $b + 6 = a$. Substituting for a, we get $b + 6 = -b \to b = -3$ and $a = 3$. So, $(a, b) = (3, -3)$.

F11B3 $\left(\frac{\mathbf{3}}{\mathbf{4\pi^2}}\right)$. The area of an equilateral triangle with side length x is $\frac{x^2\sqrt{3}}{4}$, and the area of a semicircle with radius x is $\frac{1}{2}\pi x^2$. We are given that $\frac{s^2\sqrt{3}}{4} = \frac{1}{2}\pi r^2$ and we want to compute $\frac{r^2\sqrt{3}}{4}\Big/\frac{1}{2}\pi s^2$. Solve the first equation for $\frac{r^2}{s^2}$ to obtain $\frac{r^2}{s^2} = \frac{\sqrt{3}}{2\pi}$. Then $\frac{r^2\sqrt{3}}{4}\Big/\frac{1}{2}\pi s^2 = \frac{r^2}{s^2}\cdot\frac{\sqrt{3}}{2\pi} = \frac{\sqrt{3}}{2\pi}\cdot\frac{\sqrt{3}}{2\pi} = \frac{3}{4\pi^2}$.

F11B4 $(\mathbf{2\sqrt{2}})$. Applying $\log_6$ to both sides of the equation yields

$$\begin{aligned}
\log_6\left(2^{\log_4 6}\right) &= \log_6\left(6^{\log_8 x}\right)\\
(\log_4 6)(\log_6 2) &= \log_8 x\\
\log_4 2 &= \log_8 x\\
\frac{1}{2} &= \log_8 x\\
x &= 8^{\frac{1}{2}}\\
&= \sqrt{8}\\
x &= 2\sqrt{2}
\end{aligned}$$

Note the application of the power rule and chain rule for logarithms.

F11B5 **3**. The difference between consecutive terms in an arithmetic sequence is constant, so $2x^2 - 3x = x^3 - 2x^2$. Then $x^3 - 4x^2 + 3x = 0$, from which we obtain $x = 0, 1, 3$. Since the common difference must be positive, we need $2x^2 > 0$, which is satisfied when $x = 3$. The corresponding sequence is $9, 18, 27$.

F11B6 **1104**. Note that

$$\begin{aligned}
(\tan^n 60° + \cot^n 60°)^2 &= \left(\sqrt{3}^n + \frac{1}{\sqrt{3}^n}\right)^2\\
&= 3^n + \frac{1}{3^n} + 2
\end{aligned}$$

Then

$$\left\lfloor\sum_{n=1}^{6}(\tan^n 60° + \cot^n 60°)^2\right\rfloor = \left\lfloor\left(\sum_{n=1}^{6}3^n\right) + \left(\sum_{n=1}^{6}\frac{1}{3^n}\right) + \left(\sum_{n=1}^{6}2\right)\right\rfloor$$

Since $\sum_{n=1}^{x}\frac{1}{3^n}$ is always less than 1, we need not compute it.

$$\frac{3(1-3^6)}{1-3} + 6\cdot 2 = 1092 + 12 = 1104$$

So $\lfloor\sum_{n=1}^{6}(\tan^n 60° + \cot^n 60°)^2\rfloor = 1104$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 2

Fall 2011 Solutions

F11B7 **343**. If the prime factorization of n is $n = p^a q^b r^c \cdots$, where p, q, r, $\cdots$ are distinct primes, then n has $(a+1)(b+1)(c+1)\cdots$ factors. Without loss of generality, assume that the primes are ordered so that $a \geq b \geq c \geq \cdots$. Note that $n^3 = p^{3a}q^{3b}r^{3c}\cdots$, so n^3 has $(3a+1)(3b+1)(3c+1)\cdots$ factors. If n^3 has 10 factors, then $10 = (3a+1)(3b+1)(3c+1)\cdots$. There are two ways to factor 10 into a product of positive integers greater than 1: $5\cdot 2$ and 10. From $10 = 5\cdot 2$, we obtain $3a+1 = 5$ and $3b+1 = 2$, for which there are no integer solutions for a and b. So $10 = 3a+1$. Then $a = 3$, and n must be the cube of a prime number. The largest cube of a prime less than 1000 is $7^3 = 343$.

F11B8 $\mathbf{\frac{5}{9}}$. Let $[XYZ]$ denote the area of triangle XYZ, and let $[XYZW]$ denote the area of quadrilateral $XYZW$. Note that $[ABCD] = [PQRS] + [APS] + [BQP] + [CRQ] + [DSR]$. Compare $[ABD]$ with $[APS]$. $[ABD] = \frac{1}{2}(AB)(DA)\sin A$ while $[APS] = \frac{1}{2}(AP)(SA)\sin A = \frac{1}{2}\left(\frac{2}{3}AB\right)\left(\frac{1}{3}DA\right) = \frac{2}{9}[ABC]$. Since $[ABC] = \frac{1}{2}[ABCD]$, $[APS] = \frac{1}{9}[ABCD]$. Similarly, $[BQP] = [CRQ] = [DSR] = \frac{1}{9}[ABCD]$. Then $[ABCD] = [PQRS] + 4\left(\frac{1}{9}[ABCD]\right)$. Solving for $[PQRS]$, we obtain $[PQRS] = \frac{5}{9}[ABCD]$.

F11B9 **152.** Let $s, t,$ and u be the roots of the equation. Since s, t, u is a geometric sequence, let $s = \alpha$, $t = \alpha r$, and $u = \alpha r^2$.

In the equation $ax^3 + bx^2 + cx + d = 0$, where q_1, q_2, and q_3 are the roots, $-\frac{b}{a} = q_1 + q_2 + q_3$, $\frac{c}{a} = q_1q_2 + q_2q_3 + q_3q_1$, and $-\frac{d}{a} = q_1q_2q_3$.

So, $\frac{38}{3} = \alpha + \alpha r + \alpha r^2$, $\frac{c}{3} = (\alpha)(\alpha r) + (\alpha r)(\alpha r^2) + (\alpha r^2)(\alpha)$, and $\frac{192}{3} = (\alpha)(\alpha r)(\alpha r^2)$. Then $c = 3(\alpha^2 r + \alpha^2 r^3 + \alpha^2 r^2) = 3\alpha r(\alpha + \alpha r + \alpha r^2)$. From $\frac{192}{3} = \alpha^3 r^3$, we obtain $4 = \alpha r$. Then $c = 3(4)\left(\frac{38}{3}\right) = 152$.

F11B10 $\mathbf{\frac{5\sqrt{3}}{2}}$**.** By the Law of Sines, $\frac{AC}{\sin B} = \frac{BC}{\sin A}$. So, $BC = \frac{\sin A}{\sin B}AC = \frac{\left(\sqrt{2}/2\right)}{\left(\sqrt{3}/2\right)}\left(\sqrt{15}\right) = \sqrt{10}$. Since $\overline{AB}$ is a diameter of the circle containing points A, B, and E, the angle $\angle AEB$ is a right angle. Then $\overline{DE}$ is an altitude drawn to the hypotenuse of right triangle ΔAEB. By similar triangles formed by an altitude to the hypotenuse of a right triangle, its length is the geometric mean of the lengths of the two segments of the hypotenuse, i.e., $(DE)^2 = (AD)(DB)$. In right triangle ΔADC, $\overline{AD}$ is a leg, so $AD = AC\cos A = \sqrt{15}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{30}}{2}$. Similarly, in right triangle BCD, $BD = BC\cos B = \sqrt{10}\cdot\frac{1}{2} = \frac{\sqrt{10}}{2}$. Then $(DE)^2 = \frac{\sqrt{30}}{2}\frac{\sqrt{10}}{2} = \frac{5\sqrt{3}}{2}$.

F11B11 **3982**. First, consider the numbers in the interval between 2000 and 2010, inclusive. These numbers each have one 2, two 0s, and one other digit, denoted as d. A number with the desired property in this interval must have $2 = 0 + 0 + d$, $0 = 0 + 2 + d$, or $d = 0 + 0 + 2$. Then $d = 2$ is the only solution, and 2002 is the next largest number with the desired property. Then consider the interval between 1990 and 1999. These numbers each have one 1, two 9s, and one other digit, denoted as d. However, the equations $1 = 9 + 9 + d$, $9 = 1 + 9 + d$, $d = 1 + 9 + 9$ do not have solutions if d is a digit. Finally consider the interval between 1980 and 1989. The digit 9 is the largest among the four digits, so it must be the sum of the other three. Two of the remaining three digits are 1 and 8, so the final digit is 0, and 1980 is the next largest number with the desired property. $2002 + 1980 = 3892$.

F11B12 **91.** The polynomial $x^4 + 4x^3 + 6x^2 + 1004x + 1001$ looks very similar to $x^4 + 4x^3 + 6x^2 + 4x + 1 = (x+1)^4$. In fact,

$$\begin{aligned} x^4 + 4x^3 + 6x^2 + 1004x + 1001 &= (x^4 + 4x^3 + 6x^2 + 4x + 1) + (1000x + 1000) \\ &= (x+1)^4 + 1000(x+1) \\ &= (x+1)((x+1)^3 + 1000) \\ &= (x+1)((x+1)^3 + 10^3) \\ &= (x+1)\big((x+1) + 10\big)((x+1)^2 - 10(x+1) + 10^2) \\ &= (x+1)(x+11)(x^2 + 2x + 1 - 10x - 10 + 100) \\ &= (x+1)(x+11)(x^2 - 8x + 91) \end{aligned}$$

The nonreal roots of the equation must be roots of $x^2 - 8x + 91 = 0$, so their product is 91.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 3

Fall 2011 Solutions

F11B13 **80**. Let a be Al's current test average, and let n be the number of tests Al has taken so far. From the give information, $\frac{an+90}{n+1} = a + 1$ and $\frac{an+100}{n+1} = a + 2$. Subtracting the two equations, obtain $\frac{10}{n+1} = 1$, so $n = 9$. Substituting into the first equation, $\frac{9a+90}{10} = a + 1$, from which we obtain $a = 80$.

F11B14 **30**. Substitute $x = -3$ and $x = 2$ into the functional equation.

$$f\left((-3)^2 + (-3)\right) = (-3+1)\left((-3)^3 + f(-3)\right)$$
$$f(6) = (-2)\left(-27 + f(-3)\right)$$
$$f(6) = 54 - 2f(-3)$$
$$f\left((2)^2 + (2)\right) = (2+1)\left((2)^3 + f(2)\right)$$
$$f(6) = (3)\left(8 + f(2)\right)$$
$$f(6) = 24 + 3f(2)$$

Equating the two equations for $f(6)$ yields $2f(-3) + 3f(2) = 30$.

F11B15 $\mathbf{\frac{\sqrt{2}}{6}}$. Let Q be the projection of P onto plane $ABCD$. By symmetry, it must be the rectangle's center. Then $QA = \frac{1}{2}\sqrt{\left(\sqrt{2}\right)^2 + 1^2} = \frac{\sqrt{3}}{2}$. By the Pythagorean theorem in ΔPQA, $PQ = \sqrt{1^2 - \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}$. Then the volume is $\frac{1}{3}\cdot\left(1\cdot\sqrt{2}\right)\cdot\frac{1}{2} = \frac{\sqrt{2}}{6}$.

F11B16 $\mathbf{\frac{7\sqrt{2}}{2}}$. Note that ΔABC is a right triangle, with $\angle C = 90°$. Then $\angle ACP = \angle BCP = 45°$. Since $\angle ACP$ and $\angle ABP$ intercept arc AP, they are congruent and $\angle ABP = 45°$. Similarly, $\angle BAP = 45°$. Then ΔABP is an isosceles right triangle and $AP = BP = \frac{5\sqrt{2}}{2}$. By Ptolemy's theorem, $AC \cdot BP + AP \cdot BC = CP \cdot AB$, so $3 \cdot \frac{5\sqrt{2}}{2} + 4 \cdot \frac{5\sqrt{2}}{2} = CP \cdot 5$. Then $CP = \frac{7\sqrt{2}}{2}$.

F11B17 **42**. Since $\sqrt{20000} = \sqrt{10000}\cdot\sqrt{2} = 100\cdot\sqrt{2} \approx 141.4$, the perfect sqares between 10000 and 20000 are $\{100^2, 101^2, 102^2, \ldots, 141^2\}$. So there are $141 - 100 + 1 = 42$ perfect squares between 10000 and 20000 inclusive.

F11B18 **2992**. The organized sequences can contain one, two, three, or four different numbers, and these numbers must appear consecutively. Consider the organized sequences consisting of exactly one number. There is one sequence for each number, so there are four altogether. Next, consider the sequences with two numbers; there are ${}_4P_2$ ways to choose the two numbers. With the numbers chosen, count the organized sequences by using the balls-and-urns method. There is one urn for each of the chosen numbers, and one ball for each term in the sequence. Since the urns must contain at least one ball, there are ${}_{(10+1-2)}C_1 = {}_9C_1$ ways to fill the urns. Thus, there are ${}_4P_2 \cdot {}_9C_1 = 12\cdot 9 = 108$ different organized sequences with exactly two numbers. By the same reasoning, there are ${}_4P_3 \cdot {}_9C_2 = 24\cdot 36 = 864$ organized sequences with exactly three numbers and ${}_4P_4 \cdot {}_9C_3 = 24 \cdot 84 = 2016$ organized sequences using four numbers. Altogether, there are $4 + 108 + 864 + 2016 = 2992$ organized sequences.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 4

Fall 2011 Solutions

F11B19 **5**. Let x be the shorter of CE and DE. Then $26-x$ is the longer of the two segments. Since $\overline{AB}$ and $\overline{CD}$ are intersecting chords in a circle, $AE\cdot EB=CE\cdot ED$, so $7\cdot 15=x\cdot(26-x)$. From this equation, we obtain $x=5$ or $x=21$. Since x has to be less than $26-x$, the shorter of CE and DE has to be 5.

F11B20 **12**. Note that:

$$\frac{5n-12}{n-6}=\frac{5n-30+18}{n-6}=5+\frac{18}{n-6}$$

Thus, the given expression is an integer if and only if $n-6$ is a divisor (positive or negative) of 18. Since $18=2\cdot 3^2$, it has $(1+1)(2+1)=6$ positive divisions, and so there are 12 possible values of n. (They are $-12,\ -3,\ 0,\ 3,\ 4,\ 5, 7,\ 8, 9, 12, 15,$ and 24.)

F11B21 **7**. From the triangle inequality, x must satisfy the following inequalities:

$$x^2+5x+15>9$$
$$5x+15+9>x^2$$
$$9+x^2>5x+15$$

The solution sets of these inequalities are $\{x|(x<-3) \text{ or } (x>-2)\}$, $\{x|(x>3) \text{ and } (x<8)\}$, and $\{x|(x<-1) \text{ or } (x>6)\}$ respectively. The only integer in all three sets is $x=7$, resulting in a triangle with side lengths 49, 50, and 9.
Alternatively, start listing values of x. Because $5x+15$ must be positive, we start with $x=-2$ and work our way up, reaching $x=7$ as the only integer solution to the above inequalities.

F11B22 **1**. Let $BP=x$. Then $PC=9-x$. The quadrilateral $ABPI$ can be partitioned into triangles ΔABI and ΔBIP. For both of these triangles, the height from I is equal to the radius of the incircle of triangle ΔABC; call its length r. Then the area of the triangle ΔABI is $\frac{7r}{2}$, and the area of the triangle ΔBIP is $\frac{xr}{2}$. Thus, the area of the quadrilateral $ABPI$ is $\frac{(7+x)r}{2}$. By similar reasoning, the area of the quadrilateral $AIPC$ is $\frac{(10+9-x)r}{2}$. Then $\frac{(7+x)r}{2}=\frac{(19-x)r}{2}\cdot\frac{4}{9}$. Solving for x, we obtain $x=1$.

F11B23 **0**. Since γ and $(\alpha+\beta)$ are supplementary, $\sin\gamma=\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. Then $\sin\alpha+\sin\beta=\sin\alpha\cos\beta+\cos\alpha\sin\beta$, or $(\sin\alpha)(1-\cos\beta)+(\sin\beta)(1-\cos\alpha)=0$. Each angle must be less than or equal to π because the angles are nonnegative angles that sum to π. Therefore, $\sin\alpha$ and $\sin\beta$ are nonnegative. Furthermore, α and β are less than π, since γ is at least as large as both. Suppose $\alpha>0$. Then $\sin\alpha>0$ and $1-\cos\alpha>0$. Since $\beta\geq\alpha$, we also have $\sin\beta>0$ and $1-\cos\beta>0$. Then $(\sin\alpha)(1-\cos\beta)+(\sin\beta)(1-\cos\alpha)>0$. For $\sin\alpha+\sin\beta=\sin\gamma$ to hold, α must be 0. To check that a set of such angles exists, let $\alpha=0$, $\beta=0$, and $\gamma=\pi$. Then, all 3 conditions are satisfied, since $0\leq 0\leq\pi$, $0+0+\pi=\pi$ and $\sin 0+\sin 0=\sin\pi$.

F11B24 $\mathbf{\frac{3}{10}}$. Jill can pull out three red marbles and deduce that there are two green marbles left, or she can pull out a red and a green marble (in either order) followed by a green marble and deduce that two red marbles are left. The probability of the former is $\frac{3\cdot2\cdot1}{5\cdot4\cdot3}$, and the probability of the latter is $2\cdot\frac{3\cdot2\cdot1}{5\cdot4\cdot3}$. Thus the probability that exactly 3 marbles are pulled out is $\frac{6}{60}+\frac{12}{60}=\frac{3}{10}$.

Alternatively, list all 10 possible orders for 5 draws. If r is red and g is green, the possible draw orders are: $rrrgg$, $rrgrg$, $rrggr$, $rgrrg$, $rgrgr$, $rggrr$, $grrrg$, $grrgr$, $grgrr$, and $ggrrr$. Only three of these end in three draws, namely $rrrgg$, $rggrr$, and $grgrr$, which gives a $\frac{3}{10}$ probability.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 5

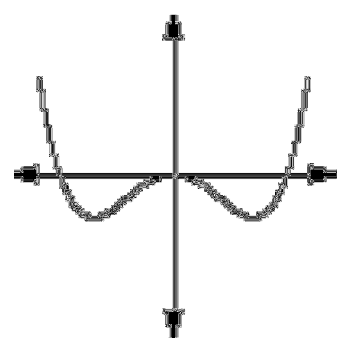
Fall 2011 Solutions

F11B25 **24**. By the pigeonhole principle, at least one of the four integers is greater than or equal to 3. Call it N. Suppose one of the other three integers is 1. The product of the four integers can be increased by replacing 1 with 2 and N with $N-1$, since $N \geq 3$ implies that $1 \cdot N < 2 \cdot (N-1)$. Thus the product can be increased if at least one of our four integers is 1. To maximize the product of the four ingeters, none of the integers can be 1. If none of the integers is 1, then the four integers must be 2, 2, 2, and 3, whose product is 24.
Alternatively, list the six possible cases: 1116, 1125, 1134, 1224, 1233, and 2223, with 2223 having the greatest product.

F11B26 $\frac{\mathbf{107}}{\mathbf{17}}$. Let X be the intersection of $\overline{PQ}$ and $\overline{AB}$. Since $\angle A$ and $\angle B$ are formed by a radius and a tangent, they are right angles. By angle-angle similarity (right angles and vertical angles), $\Delta PAX \sim \Delta QBX$ with a ratio of similitude of $\frac{PA}{QB} = \frac{1}{2}$. Then $PX = \frac{17}{3}$ and $XQ = \frac{34}{3}$. Py the Pythagorean Theorem, $AX = \frac{8}{3}$, so $BX = \frac{16}{3}$. Then $AM = 4$ and $MX = \frac{4}{3}$. Again, by angle-angle similarity, $\Delta PAX \sim \Delta MNX$ with a ratio of similitude of $\frac{PX}{MX} = \frac{17}{4}$. Then $NX = \frac{4}{17}AX = \frac{32}{51}$. Then $PN = PX + XN = \frac{17}{3} + \frac{32}{51} = \frac{321}{51} = \frac{107}{17}$.

F11B27 **6**. Let $z = -\sqrt{2} + i\sqrt{6}$. The magnitude of z is $2\sqrt{2}$, and the argument is $\tan^{-1}\frac{\sqrt{6}}{-\sqrt{2}} = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$ radians. Thus, $z^n = \left(2\sqrt{2}\right)^n e^{-\frac{n\pi}{3}i}$. If z^n is a real number, its argument $-\frac{n\pi}{3}$ must be an integer multiple of π, sp n is a multiple of 3. If z^n is an integer, its magnitude $\left(2\sqrt{2}\right)^n$ must be a whole number, so n must be an even number. The smallest positive even multiple of 3 is 6, which gives $z^6 = 512$.

F11B28 **81**. The equation $x^4 - 18x^2 = 0$ has three roots: 0, $\sqrt{18}$, and $-\sqrt{18}$, with a double root at 0. The leading coefficient is positive, so the graph "opens" upwards. Note that the function $y = x^4 - 18x^2 + b$ is an even function, so its graph will have line symmetry across y-axis. A sketch of this is to the right. As b increases, the graph of $y = x^4 - 18x^2 + b$ will have four x-intercepts, until the two local minima reach the x-axis. Then the value of b we want will force the graph of $y = x^4 - 18x^2 + b$ to have two double roots. Then $y = x^4 - 18x^2 + b$ can be factored as $(x-r)^2(x+r)^2$. Note that the roots must be additive inverses due to the line symmetry of the graph. Then $y = x^4 - 18x^2 + b = (x^2 - r^2)^2$. By completing the square, $r = 3$ and $b = 81$.



F11B29 $\mathbf{8\pi}$. Let r be the radius of the smallest circle. Then πr^2, $\pi(8)^2$, $\pi(8r)^2$, or, πr^2, 64π, $64\pi r^2$ is a geometric sequence. Then $\frac{64\pi}{\pi r^2} = \frac{64\pi r^2}{64\pi}$, or $r^2 = 8$. The area of the smallest circle is 8π.

F11B30 **40**. Compute the initial terms of the sequence: 2011, 20, 10, 5, 25, 35, 40, 20, 10, 5, 25, 35, ... Since $a_6 = a_{12}$, we know that $a_{13} = f(a_{12}) = f(a_6) = a_7$ and so forth. In fact, for any integer $k > 0$, $a_{6k} = a_6$. Then $a_{2010} = a_6$ and $a_{2011} = a_7 = 40$.