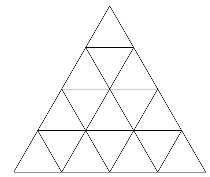
New York City Interscholastic Math League
Senior Division A Contest Number 1

Part I Spring 2011 Contest 1 Time: 10 Minutes

S11A1 Compute the number of upwards-pointing equilateral triangles whose edges belong to the segments shown.



S11A2 The real number r is larger than $1,000,000,000$. Compute, in terms of r, the median of the set $\left\{(\log_2 r)^3, \log_2(r^3), r, \log_2(\log_2(r)), \dfrac{r}{\log_2 r}\right\}$.

Part II Spring 2011 Contest 1 Time: 10 Minutes

S11A3 Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2^n - 1}{3^{n-1}}$.

S11A4 Given any point P, point $R_1(P)$ is defined to be the reflection of P through the line $y = x \cdot \tan(15°)$. Similarly, given any point P, the point $R_2(P)$ is defined to be the reflection of P through the line $y = x \cdot \tan(20°)$. Let Q be the point $(6, 0)$. Compute $R_2(R_1(R_2(R_1(R_2(R_1(Q))))))$.

Part III Spring 2011 Contest 1 Time: 10 Minutes

S11A5 Compute the smallest positive integer n such that 360^n has more than 1000 positive integral divisors.

S11A6 A cylinder of radius 1 has its axis parallel to the z-axis, and is cut by the plane $z = 4x$. The intersection of these surfaces is an ellipse. Compute the distance between its foci.

New York City Interscholastic Math League
Senior Division A — Contest Number 2

Part I — Spring 2011 — Contest 2 — Time: 10 Minutes

S11A7 Compute $\dfrac{1-\cos 34^\circ}{1+\cos 34^\circ} - \sec^2 17^\circ$.

S11A8 Compute, as a function of the positive integer n, the number of points with integer coordinates that lie inside or on the boundary of the triangle whose sides have equations $y = 0$, $y = \frac{x}{3}$ and $x = 3n$.

Part II — Spring 2011 — Contest 2 — Time: 10 Minutes

S11A9 Compute the largest coefficient in the expansion of $(2x + 3y)^6$.

S11A10 Three unit circles are pairwise tangent. There are two circles that are tangent to all three of these circles, one with radius r and the other with radius R, such that $r < R$. Compute $\frac{R}{r}$.

Part III — Spring 2011 — Contest 2 — Time: 10 Minutes

S11A11 Compute the number of positive integers less than 2011 that are divisible by 3 or divisible by 5 but not divisible by both 3 and 5.

S11A12 Define $t(n)$ by $t(0) = 1$ and for $n \geq 1$, $t(n) = 2^{t(n-1)}$. Compute the smallest positive integer n such that $t(n) \geq 4000^{(4000^{4000})}$.

New York City Interscholastic Math League
Senior Division A — Contest Number 3

Part I | Spring 2011 | Contest 3 | Time: 10 Minutes

S11A13 There are 5280 feet in a mile, 8 furlongs in a mile, 6 feet in a fathom and 40 rods in a furlong. Compute the number of square fathoms in a square rod.

S11A14 Point P lies on the circle with center $(-2,-1)$ and radius 2, while point Q lies on the circle with center $(3,-2)$ and radius 3. Compute the smallest possible distance between points P and Q.

Part II | Spring 2011 | Contest 3 | Time: 10 Minutes

S11A15 Compute the number of ways to exactly and completely cover a 2×6 rectangle with non-overlapping 2×1 and 1×2 rectangles.

S11A16 A *diamond* is a square, one of whose sides is parallel to the line $y = x$. Compute the number of diamonds, all of whose vertices belong to the set of points (x, y) such that x, y are integers between 1 and 8, inclusive.

Part III | Spring 2011 | Contest 3 | Time: 10 Minutes

S11A17 Compute all ordered pairs (a, b) of positive integers such that $a < b$ and there exists a right triangle with legs of length a and b and hypotenuse of length 25.

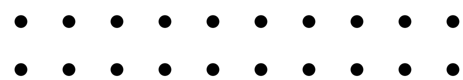
S11A18 Compute the largest coefficient in the expansion of $(x + \pi)^{100}$. (Leave your answer in the form of a power of π times a binomial coefficient.)

New York City Interscholastic Math League
Senior Division A Contest Number 4

Part I Spring 2011 Contest 4 Time: 10 Minutes

S11A19 Compute the number of positive integers n such that there exist positive integers x and y satisfying $\text{lcm}(x, y) = 44100$ and $\gcd(x, y) = n$.

S11A20 Three distinct points are chosen from a 2×10 rectangular grid of points like the one shown. Compute the probability that these points are the vertices of a nondegenerate triangle.



Part II Spring 2011 Contest 4 Time: 10 Minutes

S11A21 Compute all θ such that $0 \leq \theta \leq 360$ and $\cos 3\theta^\circ = \cos 51^\circ$.

S11A22 A bullseye is formed by seven concentric circles with radii 1, 2, 3, 4, 5, 6 and 7. A dart is thrown at the bullseye and lands at a random point inside the largest circle. Compute the expected number of the seven circles inside which the dart lands.

Part III Spring 2011 Contest 4 Time: 10 Minutes

S11A23 Suppose that the real part of the complex number z is equal to 1 and the real part of z^2 is equal to -2. Compute the real part of z^3.

S11A24 Compute, as a function of the positive integer n, the number of points with integer coordinates contained inside or on the boundary of the triangle whose sides have equations $y = 3x$, $y = \frac{x}{2}$ and $y = -\frac{x}{2} + 14n$.

New York City Interscholastic Math League
Senior Division A   Contest Number 5

Part I   Spring 2011   Contest 5   Time: 10 Minutes

S11A25 Compute the largest possible value of $2x^2\sqrt{5} - x^4$ as x varies through the real numbers.

S11A26 Compute the number of ways in which a nonnegative integer may be written in each square of a 3×3 grid so that the sum of the entries in each row and each column is equal to 2.

Part II   Spring 2011   Contest 5   Time: 10 Minutes

S11A27 A jar contains blue and green balls (and no others). If a single ball is selected at random from the jar, the probability that it is blue is $\frac{1}{3}$. If two balls are selected from the jar without replacement, the probability that they are both blue is $\frac{2}{21}$. Compute the total number of balls in the jar.

S11A28 Eight points lie on a line. Compute the number of ways in which four line segments can be drawn such that each of the points is an endpoint of exactly one of the segments and each pair of segments overlap.

Part III   Spring 2011   Contest 5   Time: 10 Minutes

S11A29 In rhombus $ABCD$, points P and Q are chosen on $\overline{AB}$ and $\overline{CD}$, respectively, so that $\frac{AP}{PB} = \frac{CQ}{QD} = \frac{1}{2}$. If $\overline{PQ} \perp \overline{AB}$, compute $\sin \angle ABC$.

S11A30 Three points, A, B and C, lie on the sphere of radius 1 centered at O. If the shortest path from A to B along the surface of the sphere has length $\pi/4$ and the shortest path from B to C along the surface of the sphere has length $\pi/4$ and the planes AOB and BOC are perpendicular, compute the length of the shortest path from A to C along the surface of the sphere.

New York City Interscholastic Math League
Senior A Division — Contest Number 1 Solutions

S11A1. **20**. There are ten upwards-pointing triangles of side-length 1, six of side-length 2, three of side-length 3 and one of side-length 4 for a total of 20 triangles.

Challenge: what if the original triangle of side-length 4 were replaced by a triangle of side-length n?

S11A2. $(\log_2 \mathbf{r})^3$. One approach is to choose a convenient value of r to use. For example, a large power of 2 will do the trick, say $r = 2^{32}$. Then $r/\log_2 r = 2^{27}$, $(\log_2 r)^3 = 32^3$, $\log_2(r^3) = 32 \cdot 3$ and $\log_2 \log_2 r = 5$. We see that the median is $32^3 = (\log_2 r)^3$.

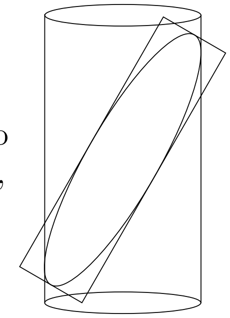
To prove this is somewhat more involved. Since $r > 4$, we have $\log_2 r > 2$ and thus $\log_2(r^3) < (\log_2 r)^3$ and $r/\log_2 r < r$. One can show by induction that $\log_2 x < x$ for all positive integers x, and with more sophisticated arguments can extend this to all positive real numbers. It follows that $\log_2(\log_2 r) < \log_2 r$ and so $\log_2(\log_2 r) < \log_2(r^3)$. Thus, to show that $(\log_2 r)^3$ is the median, we must only show that $(\log_2 r)^3 < r/\log_2 r$, or equivalently that $\log_2 r < r^{1/4}$. In fact, this inequality is valid for any $r > 2^{16}$, and can be proven via the same methods that show $\log_2 r < r$ for $r > 0$.

S11A3. **9/2**. We have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^n - 1}{3^{n-1}} &= \sum_{n=1}^{\infty} 2 \cdot (2/3)^{n-1} - \sum_{n=1}^{\infty} (1/3)^{n-1} \\ &= \frac{2}{1-2/3} - \frac{1}{1-1/3} \\ &= 9/2.\end{aligned}$$

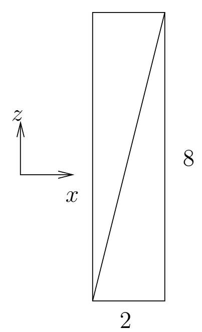
S11A4. $(\mathbf{3\sqrt{3}, 3})$. Since both lines pass through the origin, neither reflection changes the distance of any point from the origin. Thus, it is enough to keep track of the *argument* of our point, i.e., the angle at which it lies above the x-axis as we perform the reflections. Applying R_1 to a point with argument θ yields a point with argument $30^\circ - \theta$, while applying R_2 to a point with argument θ yields a point with argument $40^\circ - \theta$. Thus, applying first R_1, then R_2 to a point with argument θ yields a point with argument $40^\circ - (30^\circ - \theta) = \theta + 10^\circ$, i.e., this combination is a rotation by 10° counter-clockwise around the origin. We are asked to apply this combined operation three times to the point $(6, 0)$, which yields a point at distance 6 from the origin and argument 30°. This point is $(3\sqrt{3}, 3)$.

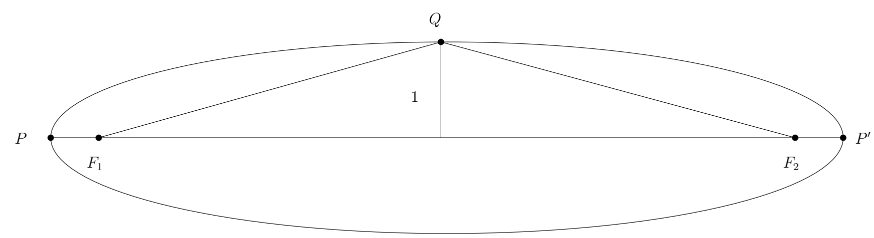
S11A5. **5**. We have $360 = 2^3 \cdot 3^2 \cdot 5$, so $360^n = 2^{3n} \cdot 3^{2n} \cdot 5^n$ has $(3n+1)(2n+1)(n+1)$ factors. Now it suffices to check a few reasonable values of n; we see that when $n = 4$ we have $13 \cdot 9 \cdot 5 = 585$ factors and when $n = 5$ we have $16 \cdot 11 \cdot 6 = 1056$ factors, so the desired value is 5.



S11A6. **8**. The major axis of the ellipse is the segment that joins the two points on the ellipse that are as far as possible from each other. In our case, one of these is the point on the ellipse with the largest x-coordinate (and also the largest z-coordinate) and the other is the point with the smallest x-coordinate (and also the smallest z-coordinate). This segment is the hypotenuse of a right triangle whose base is the diameter of the ellipse, so its length is $\sqrt{2^2 + (4 \cdot 2)^2} = 2\sqrt{17}$. The minor axis is the diameter of the ellipse perpendicular to the major axis and has length 2.

Let F_1 and F_2 be the two foci, let P and P' be the endpoints of the major axis and let Q be a point at one end of the minor axis. By the defining property of an ellipse we have $F_1P + F_2P = F_1Q + F_2Q$. From the symmetry of the ellipse we have $F_2P = F_1P'$ and so $F_1P + F_2P = PP' = 2\sqrt{17}$. By symmetry and the Pythagorean Theorem we have $F_2Q = F_1Q = \sqrt{1^2 + (F_1F_2/2)^2}$. Putting all this information together gives $2\sqrt{17} = 2\sqrt{1^2 + (F_1F_2/2)^2}$, whence $F_1F_2 = 8$. (This paragraph is a derivation of a particular case of the general fact that in an ellipse with major axis of length A, minor axis of length B and distance C between the foci, we have $C^2 = A^2 - B^2$.)





Challenge: how far can you generalize this? What about a plane $z = kx$ for some constant k? What if the radius of the cylinder changes? What about a plane $z = ax + by + c$ for constants a, b, c?

S11A7. $-\mathbf{1}$. Use the double-angle identity $\cos(2t) = 2\cos^2 t - 1$ to write both occurrences of $\cos 34°$ in terms of $\cos 17°$, and write $\sec^2 17° = \frac{1}{\cos^2 17°}$. This simplifies our expression to $\frac{2-2\cos^2 17°}{2\cos^2 17°} - \frac{1}{\cos^2 17°}$. Combining the fractions, all trig functions cancel and we are left with the answer -1. (The question arose by beginning with the trig identity $\sec^2\theta - \tan^2\theta = 1$, performing several manipulations, and choosing the irrelevant θ to be $17°$.)

S11A8. $(\mathbf{3n^2+5n+2})/\mathbf{2}$ OR $\mathbf{3n^2/2+5n/2+1}$ OR $(\mathbf{3n+2})(\mathbf{n+1})/\mathbf{2}$, or equivalent. There are exactly k lattice points in the triangle with x-coordinate $3k-3$, and with x-coordinate $3k-2$, and with x-coordinate $3k-1$. Thus, the total number of lattice points in the triangle is $(n+1) + 3\sum_{k=1}^{n} k = (n+1) + 3\cdot\frac{n(n+1)}{2} = \frac{3n^2+5n+2}{2}$. (The first summand counts the right-most edge of the triangle.)

Alternatively, use Pick's Theorem: the area of the polygon is $\frac{3n^2}{2}$, and this should be equal to $I + \frac{B}{2} - 1$, where I is the number of interior points and B is the number of boundary points. The horizontal leg of the triangle has $3n+1$ points, the vertical leg has $n+1$ and the hypotenuse has $n+1$, so (accounting for double-counting of the vertices) $B = 5n$. It follows that $I = \frac{3n^2}{2} - \frac{5n}{2} + 1$ and so the total number of points is $I + B = \frac{3n^2}{2} + \frac{5n}{2} + 1$

S11A9. **4860**. One option is to multiply the expression in question out by hand. Alternatively, observe that the coefficient of $x^k y^{6-k}$ has the form $\frac{6!}{k!(6-k)!}2^k 3^{6-k}$. The ratio between the kth and $(k+1)$th coefficient is thus $\frac{\frac{6!}{k!(6-k)!}2^k3^{6-k}}{\frac{6!}{(k+1)!(5-k)!}2^{k+1}3^{5-k}} = \frac{3(k+1)}{2(6-k)}$. This ratio is greater than 1 when $3k+3 > 12-2k$, i.e., when $k \geq 2$, and is less than 1 when $k < 2$. Thus, starting from $k = 0$, the coefficients increase until they reach the x^2y^4 term, at which point they begin to decrease. It follows that the maximum coefficient is $\frac{6!}{2!\cdot 4!}\cdot 2^2\cdot 3^4 = 4860$.

S11A10. $\mathbf{7+4\sqrt{3}}$. We have that $1+r$ is the circumradius of an equilateral triangle of side length 2, so $1+r = \frac{2\sqrt{3}}{3}$ and thus $r = \frac{2\sqrt{3}}{3} - 1$. We also have $R = 2+r = \frac{2\sqrt{3}}{3} + 1$. Their ratio is $7+4\sqrt{3}$.

S11A11. **804**. Since 2010 is divisible by 3, the number of positive integers less than or equal to 2010 that are divisible by 3 is $\frac{2010}{3} = 670$. Similarly, the number of positive integers less than or equal to 2010 divisible by 5 is $\frac{2010}{5} = 402$. An integer is divisible by 3 and 5 if and only if it is divisible by 15, and there are $\frac{2010}{15} = 134$ integers less than or equal to 2010 that are divisible by 15. Thus, we have in total $(670-134) + (402-134) = 804$ positive integers less than 2011 that are divisible by 3 or 5 but not both.

S11A12. **6**. The answer is 6. To prove this, we can observe that $t(4) = 65536 > 4096\cdot 12 + 4 > 4000\log_2(4000) + \log_2(\log_2(4000))$. Now $t(5) = 2^{65536}$ is much smaller than $4000^{4000^{4000}}$ (since $2 < 4000$ and $65536 < 4000^{4000}$, but (exponentiating on both sides of the inequality in the previous sentence) $t(5) > 4000^{4000}\cdot\log_2(4000)$ and so (exponentiating again) $t(6) > 4000^{4000^{4000}}$.

To actually come up with this answer in the first place, one can begin by taking logarithms base 2 – taking two logarithms of $t(n) > 4000^{4000^{4000}}$ gives $t(n-2) > 4000\log_2 4000 + \log_2\log_2 4000$. Now notice that $4000 \approx 4096 = 2^{12}$, so the right-hand side is approximately $48,000$. The nearest powers of 2 are $32,768 = 2^{15}$ and $65,536 = 2^{16} = t(4)$.

Note that applying exponential functions or logarithms to both sides of an inequality preserves that inequality because these functions are both monotonically increasing. In general, the tower functon $t(n)$ grows much faster than any exponential function.

S11A13. $\frac{\mathbf{121}}{\mathbf{16}}$. We have

$$1 \text{ rod}^2 = \frac{1}{40^2} \text{ furlong}^2 = (\frac{1}{40\cdot 8})^2 \text{ mile}^2 = = (\frac{5280}{40\cdot 8})^2 \text{ foot}^2 = (\frac{5280}{40\cdot 8\cdot 6})^2 \text{ fathom}^2 = \frac{121}{16} \text{ fathom}^2.$$

(Note: these are all real units in the Imperial system of measurement used in Great Britain until the mid-1960s.)

S11A14. $\mathbf{\sqrt{26}-5}$. The centers of the two circles are at distance $\sqrt{5^2+1^2}=\sqrt{26}$. This is larger than the sum of the radii of the circles, so they do not intersect; thus, the closest the two points can be is if they lie on the segment connecting the centers of the circles, in which case they are $\sqrt{26}-5$ units apart.

S11A15. **13**. Let f_n be the number of ways to cover a $2\times n$ rectangle with dominoes. Then $f_1 = 1$ and $f_2 = 2$. Also, for any $n > 2$, the leftmost column of the rectangle can be covered by one vertical domino (leaving a $2\times(n-1)$ rectangle to be covered) or by two horizontal dominoes (leaving a $2\times(n-2)$ rectangle to be covered). Thus $f_n = f_{n-1} + f_{n-2}$, so we recognize f_n as a Fibonacci number, and in particular can compute $f_3 = 3$, $f_4 = 5$, $f_5 = 8$ and $f_6 = 13$.

Extension: What is f_0? Why?

S11A16. **56**. Consider the possible locations for the center of a diamond. If the side-length of the diamond is $d\sqrt{2}$ then the center must be one of the lattice points in the $(8-2d)\times(8-2d)$ central portion of the grid. Thus, the total number of diamonds meeting our conditions is $6^2+4^2+2^2 = 56$.

Challenge: what happens if we replace 8 by n in the statement of the problem?

S11A17. $(\mathbf{7},\mathbf{24})$ **and** $(\mathbf{15},\mathbf{20})$. Two such pairs are $(15,20)$ and $(7,24)$. One can check by a variety of methods that these are the only examples.

S11A18. $\boldsymbol{\binom{100}{24}\pi^{76}}$ or equivalent. The coefficient of x^n in $(x+\pi)^{100}$ is $\frac{100!}{n!(100-n)!}\pi^{100-n}$. Let us compare this to the coefficient $\frac{100!}{(n+1)!(99-n)!}\pi^{99-n}$ of x^{n+1} by taking their ratio,

$$\frac{\frac{100!}{n!(100-n)!}\pi^{100-n}}{\frac{100!}{(n+1)!(99-n)!}\pi^{99-n}} = \frac{(n+1)\pi}{100-n}.$$

Rearranging, we see that this ratio is less than 1 exactly when $n > \frac{100-\pi}{\pi+1}$ and is greater than 1 otherwise. Thus, the maximum coefficient occurs when $n = \lceil\frac{100-\pi}{\pi+1}\rceil$. Any reasonable approximation to π (e.g., 3.14 or even just 3.1) allows us to compute that this yields $n = 24$, so the maximal coefficient is $\binom{100}{24}\pi^{76}$.

S11A19. **81**. Certainly the gcd of two integers divides their lcm. Also, for any divisor d of 44100, we see that the pair $(x, y) = (d, 44100)$ yields a gcd equal to d. Thus, we just need to count divisors of $44100 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$, of which there are $(2+1)^4 = 81$.

S11A20. $\mathbf{\frac{15}{19}}$. The three points are the vertices of a nondegenerate triangle unless they lie on the same horizontal line. Thus, the probability in question is

$$\frac{\binom{20}{3} - 2 \cdot \binom{10}{3}}{\binom{20}{3}} = \frac{15}{19}.$$

S11A21. $\mathbf{17, 137, 257, 103, 223, 343}$. (Order is not important, but all six values must be present.) We must have either $3\theta = 51 + 360n$ for some integer n or $3\theta = -51 + 360n$ for some integer n. Taking the values of n that yield a θ in the appropriate range, we find the six possible values of θ are 17, 137, 257, 103, 223, and 343.

S11A22. $\mathbf{\frac{20}{7}}$. The probability that the dart lies inside the circle of radius i is $\frac{i^2}{7^2}$. Thus, by linearity of expectation, the desired probability is

$$\sum_{i=1}^{7} \frac{i^2}{7^2} = \frac{1}{7^2} \cdot \frac{7 \cdot 8 \cdot 15}{6} = \frac{20}{7}.$$

(To go from the first to the second term of this equality, we use the formula for the sum of consecutive perfect squares.)

Alternatively, compute using the definition of expectation: the probability that the dart lies inside exactly n circles is $\frac{(8-n)^2\pi - (7-n)^2\pi}{49\pi} = \frac{15-2n}{49}$. Therefore, the expected number of circles inside which the dart lands is $\frac{13}{49} \cdot 1 + \frac{11}{49} \cdot 2 + \ldots + \frac{3}{49} \cdot 6 + \frac{1}{49} \cdot 7 = 20/7$.

Challenge: what if there were n circles instead of 7?

S11A23. $\mathbf{-8}$. From the first given, we have $z = 1 + bi$ for some real number b. Thus $z^2 = (1 - b^2) + 2bi$. From the second given, $1 - b^2 = -2$, so $b = \pm\sqrt{3}$. It follows that $z^3 = (1 - 3b^2) + (3b - b^3)i = -8$ has real part -8.

S11A24. $\mathbf{70n^2 + 8n + 1}$. The vertices of the triangle are $A(0, 0)$, $B(14n, 7n)$ and $C(4n, 12n)$. Since these are all lattice points, we may apply Pick's Theorem. Edge $\overline{AB}$ contains $7n + 1$ lattice points, edge $\overline{AC}$ contains $4n + 1$ and edge $\overline{BC}$ contains $5n + 1$; removing the double-counted vertices, this is $16n$ boundary points. Using whatever your favorite method is, you may compute that the area of the triangle is $70n^2$. Thus, if I is the number of interior points and B is the number of boundary points, $I + B/2 - 1 = 70n^2$ and so $I = 70n^2 - 8n + 1$. Thus, the total number of lattice points on or in the triangle is $70n^2 + 8n + 1$.

Alternatively, one can split the triangle into two pieces and count the lattice points in each piece via a variation on the solution to contest 2, problem 2.

This polynomial is called the *Erhart polynomial* associated to the triangle with vertices $(0, 0)$, $(14, 7)$ and $(4, 12)$. (Note that the given triangle is a dilation of this triangle by a factor of n.) We can form the Erhart polynomial of any polygon $\mathbf{P}$ whose vertices have integer coordinates: let $E_{\mathbf{P}}(n)$ be the number of lattice points in the a dilation of $\mathbf{P}$ by a factor of n. As a challenge, try to prove the following facts: the Erhart polynomial $E_{\mathbf{P}}(n)$ of polygon $\mathbf{P}$ is a quadratic, has constant coefficient 1, and has coefficient of n^2 equal to the area of $\mathbf{P}$. What if we consider polyhedra instead of polygons?

Senior A Division Contest Number 5 Solutions

S11A25. **5**. Complete the square to write $2x^2\sqrt{5} - x^4 = 5 - (x^2 - \sqrt{5})^2$. Since $(x^2 - \sqrt{5})^2$ is always nonnegative, the largest value this expression can take is 5, when $x = \pm\sqrt[4]{5}$.

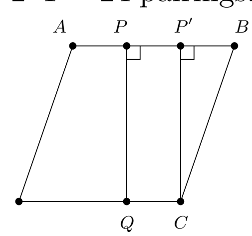
S11A26. **21**. We do a case analysis based on the number of 2s: if there are no 2s then each row contains one 0 and two 1s, and the three 0s must lie in different columns; this can occur in 6 ways. If there is one 2, it can lie in any of 9 positions; in this case, the entries in its row and column must be 0 and the other entries must be 1, so the position of the 2 determines the entire table and we haeve 9 possibilities. It is not possible to have exactly two 2s. With three 2s, we must have all other entries equal to 0, and the 2s must lie in different rows and columns. This can happen in 6 ways. This gives a total of $6 + 9 + 6 = 21$ tables meeting the desired conditions.

Note that this question ($n \times n$ tables with row and column sum equal to m) is difficult in general, and even the case $m = 2$ for arbitrary n is nontrivial. Challenge: show that if n is fixed and m varies, the number of fillings is a polynomial in m. Can you compute this polynomial for $n = 1$, $n = 2$, $n = 3$ and $n = 4$?

S11A27. **15**. Let the number of blue balls be b, so the total number of balls is $3b$. Then the probability of drawing blue twice is $\frac{b(b-1)}{3b(3b-1)} = \frac{2}{21}$. Multiplying out, this implies $7b - 7 = 6b - 2$ so $b = 5$ and there are 15 balls total.

S11A28. **24**. Call the points (in their order along the line) A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4. Suppose for sake of contradiction that one of our segments has as its endpoints two A-vertices. Then there will be two A-vertices and four B-vertices left to be connected by the other three segments. This means that one of the segments must connect two B-vertices. But the segment connecting the two A-vertices and the segment connecting the two B-vertices cannot overlap, a contradiction. Thus, each of the four segments must have one A-endpoint and one B-endpoint. In this case, all four segments necessarily intersect (e.g., they all contain the point A_4), so any such pairing works. The number of such pairings is 24: there are four points that could be connected to A_1, then three remaining points to connect to A_2, then two to connect to A_3, then one remaining point for A_4, so $4 \cdot 3 \cdot 2 \cdot 1 = 24$ pairings.

Challenge: what if there are $2n$ points connected by n segments?



S11A29. $\mathbf{\frac{2\sqrt{2}}{3}}$. Without loss of generality, set $AP = 1$, so $PB = 2$, $BC = 3$ and $CQ = 1$. Choose P' on $\overline{AB}$ so that $\overline{CP'} \perp \overline{AB}$. Then $PP' = QC = 1$ and so $BP' = 1$. Applying the Pythagorean Theorem in $\triangle BCP'$, we have $CP' = 2\sqrt{2}$ and so $\sin\angle ABC = \frac{CP'}{BC} = \frac{2\sqrt{2}}{3}$.

S11A30. $\mathbf{\frac{\pi}{3}}$. We choose a coordinate system as follows. Let O be the origin, let B be the point $(1, 0, 0)$ and let A lie in the xy-plane, with positive y-coordinate. Since the distance between A and B on the sphere is $\pi/4$ and the radius of the sphere is 1, we must have $m\angle AOB = \pi/4$ and so $A = (\sqrt{2}/2, \sqrt{2}/2, 0)$. Since planes AOB and BOC are perpendicular, C lies in the xz-plane. As before, $m\angle COB = \pi/4$ and so C has coordinates $(\sqrt{2}/2, 0, \sqrt{2}/2)$. (Actually, the z-coordinate of C could be either positive or negative, but in case it is negative we may reflect the whole picture through the xy-plane without changing anything important.)

It follows that $AC = \sqrt{0^2 + (-\sqrt{2}/2)^2 + (\sqrt{2}/2)^2} = 1$ and so $\triangle AOC$ is an equilateral triangle. Thus $m\angle AOC = \pi/3$. The distance from A to C along the sphere is the length of the arc of the circle centered at O that passes through them; since this is a unit circle, the measure of the arc is $\pi/3$.

Interesting note: this is a special case of the "Spherical Pythagorean Theorem." In general, if A, B and C lie on a sphere with center O such that planes OAB and OBC are perpendicular, then $\cos m\angle AOC = \cos m\angle AOB \cdot \cos m\angle BOC$. This can be further generalized to give a "Spherical Law of Cosines."