New York City Interscholastic Math League
Soph-Frosh Division   Contest Number 1

Part I   Fall 2010   Contest 1   Time: 10 Minutes

F10S1 Compute the product of the roots of $(3x + 4)(x - 2)(x + 3) = 0$.

F10S2 A thin pole is 7 feet tall. A storm breaks the pole at a point y feet above the ground so that the upper part, still attached to the bottom of the pole, touches the ground 2 feet away from the base. Compute y.

Part II   Fall 2010   Contest 1   Time: 10 Minutes

F10S3 The average of 30 numbers is 60. If the average of 10 of the 30 numbers is 50, compute the average of the remaining 20 numbers.

F10S4 Compute all integers n for which $\frac{n}{18-2n}$ is the square of an integer.

Part III   Fall 2010   Contest 1   Time: 10 Minutes

F10S5 The product of three consecutive positive integers is 24 times their average. Compute the sum of their squares.

F10S6 A cylinder is lying on its side. The radius of one of the circular bases and the distance between the circular bases are 6 meters and 12 meters, respectively. The cylinder is filled with water to a height of 3 meters. Compute, in terms of π, the volume of water in cubic meters.

New York City Interscholastic Math League
Soph-Frosh Division Contest Number 2

Part I Fall 2010 Contest 2 Time: 10 Minutes

F10S7 Compute the probability that a randomly selected positive factor of 24 is less than 5.

F10S8 Real numbers a and b satisfy the equations $5^a = 25^{b-3}$ and $3^{2b-2} = 27^a$. Compute $a + b$.

Part II Fall 2010 Contest 2 Time: 10 Minutes

F10S9 A square has a side length of 6. Compute the area of the region containing all points that are outside of the square but not more than two units from a point on the square.

F10S10 Compute

$$\frac{4020 \cdot (2011^2 + 2011 + 1)}{(2011)^3 - 1}$$

Part III Fall 2010 Contest 2 Time: 10 Minutes

F10S11 Suppose that $f(x+7) = 6x^2 + 3x + 4$ and $f(x) = ax^2 + bx + c$. Find $a + b + c$.

F10S12 An equiangular octagon has four sides of length 2 and four sides of length $\sqrt{2}$. All eight sides are arranged so that no two successive sides have the same length. Compute the area of the octagon.

New York City Interscholastic Math League
Soph-Frosh Division   Contest Number 3

Part I   Fall 2010   Contest 3   Time: 10 Minutes

F10S13   A student answered 5 math problems correctly. Each problem was worth either 3 or 4 points. Compute the number of different possible point totals that could be scored by the student.

F10S14   A right triangle has area 19, and the sum of its two legs is 20. Compute the length of the hypotenuse.

Part II   Fall 2010   Contest 3   Time: 10 Minutes

F10S15   Compute the units digit of $2009^{2011} + 2^4$.

F10S16   Compute the sum of the real roots of the equation

$$x^2 + 10x + 13 = 2\sqrt{x^2 + 10x + 12}$$

Part III   Fall 2010   Contest 3   Time: 10 Minutes

F10S17   Compute the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ... + \frac{1}{97 \cdot 98}$$

F10S18   A pyramid has a square base $ABCD$ with a vertex E. The area of the base is 144. The areas of $\triangle ADE$ and $\triangle BCE$ are 60 and 48, respectively. Compute the volume of the pyramid.

New York City Interscholastic Math League
Soph-Frosh Division — Contest Number 1 Solutions

F10S1. **8**. The roots of the equation are $\frac{-4}{3}$, 2, and -3, and the product of the three roots is $\left(\frac{-4}{3}\right) \cdot (2) \cdot (-3) = 8$.

F10S2. $\mathbf{\frac{45}{14}}$. The upper part of the pole has length $7-y$ and becomes the hypotenuse of a right triangle. The height is y, and the base is 2. Using the Pythagorean Theorem, we have:

$$\begin{aligned} y^2 + 2^2 &= (7-y)^2 \\ y^2 + 4 &= 49 - 14y + y^2 \\ 14y &= 45 \\ y &= \frac{45}{14} \end{aligned}$$

F10S3. **65**. The sum of all 30 numbers is $30 \cdot 60 = 1800$, and the sum of the 10 numbers is $10 \cdot 50 = 500$. Hence the sum of the remaining 20 numbers is $1800 - 500 = 1300$. The average of the 20 numbers is given by $\frac{1300}{20} = 65$.

Alternate solution. Use weighted average. The set of 20 scores has twice the weight of the set of 10 scores, so the average of the set of 20 scores must be half the distance from 60 compared to the set of 10 scores. Thus the average of the set of 20 scores is 65.

F10S4. **0, 6, 8**. We can express the problem as follows:

$$\frac{n}{18-2n} = a^2$$

where a is an integer. Rearranging the terms on the above equation to solve for n, we get:

$$\begin{aligned} n &= (18-2n) \cdot a^2 \\ n(1+2a^2) &= 18a^2 \\ n &= \frac{18a^2}{1+2a^2} \end{aligned}$$

Using long division on the above equation, we have:

$$n = 9 - \frac{9}{1+2a^2}$$

Since n and 9 are both integers, $\frac{9}{1+2a^2}$ must be an integer as well. This is an integer when $a = \pm 0, \pm 1, \pm 2$, which respectively yield $n = 0$, 6, and 8.

Alternate solution. If $n < 0$, then the numerator of $\frac{n}{18-2n}$ will be negative and the denominator will be positive, so it cannot be a square. If $n > 8$, then the numerator of $\frac{n}{18-2n}$ will be positive and the denominator will be negative, so it cannot be a square. Thus, all solutions are between $n = 0$ and $n = 8$. Substituting all these values of n, we find that $\frac{n}{18-2n}$ is a square when $n = 0$, 6, or 8.

F10S5. **77**. Let $a-1$, a, and $a+1$ be three consecutive positive integers. Then the product of the three numbers can be expressed as $(a-1)(a)(a+1) = a(a^2-1) = a^3 - a$. The sum of the three numbers is $3a$, and then their average is a. Since the product is 24 times the average, we have:

$$a^3 - a = 24a$$

Simplifying the equation results in

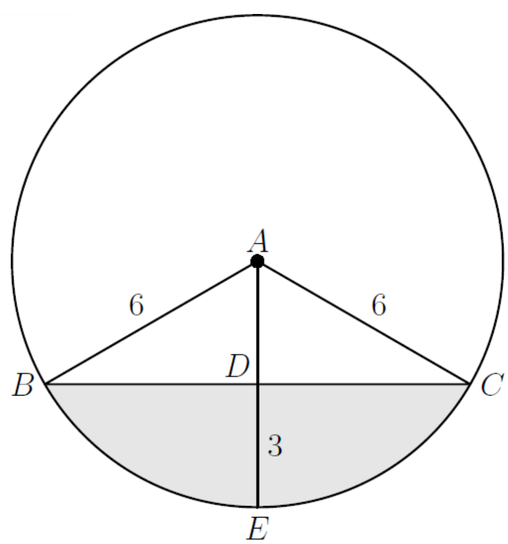
$$a^3 - 25a = 0$$

or

$$a(a+5)(a-5) = 0$$

Since a is a positive integer, $a = 5$. Then, the three consecutive integers are 4, 5, and 6, so the sum of their squares is $16 + 25 + 36 = 77$.

F10S6. $\mathbf{144\pi - 108\sqrt{3}}$. Since $DE = 3$, $AD = AE - DE = 6 - 3 = 3$. Using the Pythagorean Theorem, $DC = BD = 3\sqrt{3}$.



Since $\cos(\angle DAC) = \frac{3}{6} = \frac{1}{2}$, we see that $\angle DAC = 60°$. Therefore, $\angle BAC = 120°$. The area of sector $ABEC$ is

$$\pi(6)^2 \cdot \frac{120^\circ}{360^\circ} = 12\pi$$

In order to find the shaded area, the area of $\triangle ABC$ must be subtracted from sector $ABEC$:

$$\text{Area}BEC = 12\pi - \frac{6\sqrt{3} \cdot 3}{2} = 12\pi - 9\sqrt{3}$$

The volume of water is equal to the shaded area times the length: that is, $12(12\pi - 9\sqrt{3}) = 144\pi - 108\sqrt{3}$.

F10S7. $\mathbf{\frac{1}{2}}$. The divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24. Four of these eight divisors - 1, 2, 3, and 4 - are less than 24, so the probability that a randomly selected one is less than 5 is $\frac{4}{8} = \frac{1}{2}$.

F10S8. **6**. The first given equation can be simplified to:

$$5^a = 25^{b-3} \implies 5^a = 5^{2b-6} \implies a = 2b - 6$$

Similarly, we can simplify the second equation to

$$3^{2b-2} = 27^a \implies 3^{2b-2} = 3^{3a} \implies 2b - 2 = 3a$$

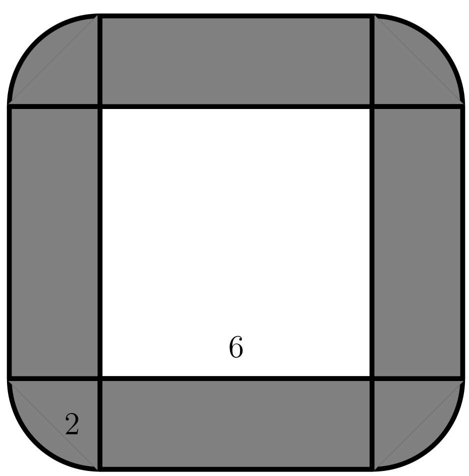
Substituting $a = 2b - 6$ into the a in $2b - 2 = 3a$, we get:

$$2b - 2 = 3(2b - 6)$$

so $b = 4$. Since $a = 2b - 6$, it follows that $a = 2$. Then, $a + b = 2 + 4 = 6$.

F10S9. $\mathbf{48 + 4}\pi$. As shown in the diagram, the area of the region (shaded) is

$$4 \cdot 12 + \pi \cdot 2^2 = 48 + 4\pi$$



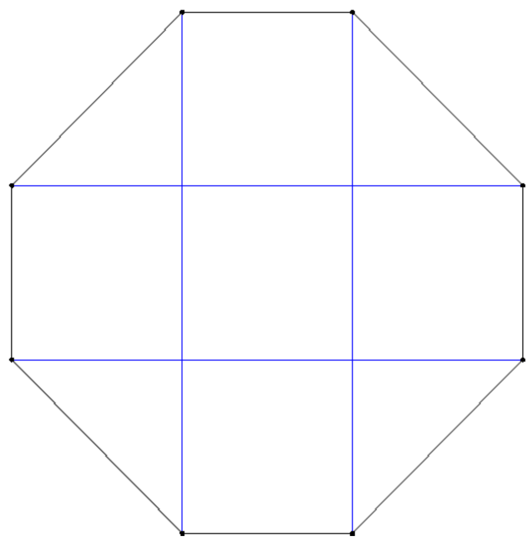
F10S10. **2**. Using the factorization $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, we can write

$$\frac{4020(2011^2 + 2011 + 1)}{2011^3 - 1} = \frac{4020(2011^2 + 2011 + 1)}{(2011-1)(2011^2 + 2011 + 1)}$$

and then cancel to write this as $\frac{4020}{2010} = 2$.

F10S11. **202**. Since $a + b + c = f(1)$, all we need to do to find the sum is substitute $x = -6$: $f(-6+7) = f(1) = 6(-6)^2 + 3(-6) + 4 = 202$.

F10S12. **14**. As shown in the diagram, the area of the octagon can be computed by adding the areas of four triangles and 5 squares.



Since the hypotenuse of each isosceles right triangle is 2, the side length of each one is $\sqrt{2}$, so the area is $\frac{\sqrt{2}\cdot\sqrt{2}}{2} = 1$. So the total area of all the triangles is $4 \cdot 1 = 4$. The area of each square is $\sqrt{2} \cdot \sqrt{2} = 2$. Since there are 5 squares, the total area of all the squares is $5 \cdot 2 = 10$. Then, the total area of the figure is $10 + 4 = 14$.

Alternate solution. The octagon is contained in a square of side length $3\sqrt{2}$. The area of the octagon is the area of the square minus the areas of four triangles. The area of the square is $(3\sqrt{2})^2 = 18$, and the area of each triangle is $\frac{\sqrt{2}\cdot\sqrt{2}}{2} = 1$, so the total area is $18 - 4(1) = 14$.

Alternate solution. Rotate the octagon by $45°$. The octagon is now contained in a square of side length 4, with 4 isosceles right triangles of side length 1 removed. Thus, the area of the octagon is $4 \cdot 4 - 4 \cdot \left(1 \cdot 1 \cdot \frac{1}{2}\right) = 16 - 2 = 14$.

New York City Interscholastic Math League
Soph-Frosh Division — Contest Number 3 Solutions

F10S7. **6**. Let a positive integer x be the number of problems that were worth 3 points. Then, $5-x$ problems were worth 4 points. The total number of points, y, can be expressed as $3x+4(5-x)$, which can be rewritten as $20-x$. Since x ranges from 0 (i. e. all five problems were worth 4 points) to 5 (all five problems were worth 3 points), the number of different points possible is 6: $20, 19, 18, 17, 16, 15$.

Alternate solution. There could be either 0, 1, 2, 3, 4, or 5 problems worth 3 points. Each one of these will yield exactly one point total, so the total number of scores is 6.

F10S8. **18**. Let a and b be the two legs of the triangle. Then $a+b=20$ and $\frac{ab}{2}=19$. The length of the hypotenuse is $\sqrt{a^2+b^2}=\sqrt{(a+b)^2-2ab}=\sqrt{(20)^2-2(38)}=\sqrt{324}=18$.

F10S9. **5**. For 2009, the units digit cycles in a cycle length of two: $9, 1, 9, 1, ...$ If the power of 2009 is an even number, then the units digit is one, and if the power is odd then the units digit is 9. Since here the power (2011) is odd, the units digit is 9. The units digit of $2^4=16$ is 6. Then, the units digit of the sum is the units digit of $9+6=15$, which gives the answer 5.

F10S10. $\mathbf{-10}$. First, let $y=\sqrt{x^2+10x+12}$. Then, the equation can be simplified to $y^2+1=2y$, or $y^2-2y+1=0$, or $(y-1)^2=0$, so $y=1$. Thus, we have $1=\sqrt{x^2+10x+12}$, so $x^2+10x+11=0$. Now, one path is to explicitly find the roots using the quadratic formula. They are $-5+\sqrt{14}$ and $-5-\sqrt{14}$, so the sum is -10. We could also use Vieta's formula, which tells us that the sum of the roots is the negative of the x coefficient of the polynomial - that is, -10.

F10S11. $\mathbf{\frac{97}{98}}$. Each term can be rewritten as follows:

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+...+\left(\frac{1}{97}-\frac{1}{98}\right)$$

By associativity, this is equal to

$$1+\left(\frac{1}{2}-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{3}\right)+...+\left(\frac{1}{97}-\frac{1}{97}\right)-\frac{1}{98}=\frac{97}{98}$$

F10S12. $120\sqrt{7}$. Since the square base is of area 144, its side length is $\sqrt{144} = 12$. Using the formula $A = \frac{1}{2}bh$ (A is the area, b and h are the base and height respectively), the heights (altitudes) of $\triangle ADE$ and $\triangle BCE$ are $\frac{2 \cdot 60}{12} = 10$ and $\frac{2 \cdot 48}{12} = 8$ respectively. Let H be the height of the pyramid and x be the distance from the point where H intersects the base to $\bar{AD}$. Since x, H, and the height of $\triangle ADE$ form a right triangle, we have:

$$10^2 - H^2 = x^2$$

Similarly, $(12 - x)$, H, and the height of $\triangle BCE$ form a right triangle:

$$8^2 - H^2 = (12 - x)^2$$

Subtracting these equations, we get

$$10^2 - 8^2 = x^2 - (12 - x^2)$$

or

$$36 = 24x - 144$$

or $x = \frac{15}{2}$, which means that $H = \frac{5\sqrt{7}}{2}$. The volume of the pyramid is $\frac{1}{3}BH$, where B is the area of the base of the triangle. This is $\frac{1}{3}(144)(\frac{5\sqrt{7}}{2})$, or $120\sqrt{7}$.