New York City Interscholastic Mathematics League

Junior Division   Contest Number 1

Part I   ***Spring 2010***   ***Contest 1***   ***Time: 10 Minutes***

S10J1 The numbers 72, 8, 24, 10, 5, 45, 36, 15 are grouped into four pairs so that the product of the numbers in each pair is the same. Compute that product.

S10J2 In triangle ABC, $AB = 5$, $BC = 3$, $AC = 4$, and I is the center of the inscribed circle. Compute the number of degrees in the measure of $\measuredangle AIB$.

Part II   ***Spring 2010***   ***Contest 1***   ***Time: 10 Minutes***

S10J3 A 5-by-7 rectangle is divided into 35 1-by-1 squares. A line segment joining two opposite vertices is drawn. Compute the number of 1-by-1 squares whose interiors are intersected by the line segment.

S10J4 Compute $\sqrt{7+\sqrt{1+\sqrt{7+\sqrt{1+\cdots}}}}$.

Part III   ***Spring 2010***   ***Contest 1***   ***Time: 10 Minutes***

S10J5 Compute the number of subsets of {1, 2, 3, ..., 10} that consist entirely of two or more consecutive integers.

S10J6 Two circles intersect at A and B, $A \neq B$. A line is tangent to one circle at P and the other at Q. Line AB intersects $\overline{PQ}$ at T. If the radii of the circles are 3 and 5, and the distance between their centers is 6, compute PT.

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Junior Division CONTEST NUMBER 2

PART I ***SPRING 2010*** ***CONTEST 2*** ***TIME: 10 MINUTES***

S10J7 A 5-by-5 square is divided into 25 1-by-1 squares. Compute the number of sets of two 1-by-1 squares such that neither square is in the same row or column as the other square.

S10J8 Set S is a subset of $\{1, 2, 3, ..., 15\}$, and the sum of the elements of S is 110. Let n be the number of elements of S. Find all possible values of n.

PART II ***SPRING 2010*** ***CONTEST 2*** ***TIME: 10 MINUTES***

S10J9 When air is pumped into a certain balloon, the volume of the balloon doubles every second until it pops when it obtains a volume of 800 cubic centimeters. It takes exactly 10 seconds from the time air first begins to be pumped into the balloon until it pops. After how many seconds from the time air first begins to be pumped into the balloon will the volume of the balloon be 400 cubic centimeters?

S10J10 Find both ordered pairs (a, b) of positive integers such that $2 = \sqrt{a + \sqrt{b + \sqrt{a + \sqrt{b + \cdots}}}}$.

PART III ***SPRING 2010*** ***CONTEST 2*** ***TIME: 10 MINUTES***

S10J11 Find the smallest integer $N > 1$ such that there exist positive integers a and b satisfying $N = a^4 = b^6$.

S10J12 In rectangle $ABCD$, $AB = 6$ and $BC = 4$. The circle with diameter $\overline{AB}$ meets the circle with diameter $\overline{CD}$ at E and F. Compute EF.

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Junior Division CONTEST NUMBER 3

PART I ***SPRING 2010*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S10J13 Solve for x: $\left(4^4\right)^x = 8^8$.

S10J14 Define a *closed polygonal path* as the union of three or more line segments satisfying the following conditions:

- Each line segment shares each one of its endpoints with exactly one other line segment.
- Each endpoint is the intersection of only two line segments.
- No subset of the line segments forms another closed polygonal path.

Given six points on a circle, compute the number of closed polygonal paths that include all these points as vertices.

PART II ***SPRING 2010*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S10J15 The integers from 1 to 10, inclusive, are listed in a row in order. In each of the nine spaces between adjacent pairs of numbers, a plus sign or a minus sign is inserted. Find the number of assignments of plus and minus signs that will yield an expression whose value is 0.

S10J16 Two externally tangent circles are inside square $ABCD$, and $AB = 1$. One of the circles is tangent to $\overline{AB}$ and $\overline{AD}$, and the other is tangent to $\overline{BC}$ and $\overline{CD}$. The radius of one circle is twice that of the other. Find the sum of the radii of the two circles.

PART III ***SPRING 2010*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S10J17 Compute the number of sequences of letters that consist of at least one letter and at most 4 letters that can be made using the letters from the set $\{A, B, C\}$.

S10J18 Find the smallest positive integer n such that $10n$ is a perfect square and $6n$ is a perfect cube.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division CONTEST NUMBER 1

Spring 2010 Solutions

S10J1 **360**. Factor the eight numbers into primes: $2^3{\bullet}3^2$, 2^3, $2^3{\bullet}3$, $2{\bullet}5$, 5, $3^2{\bullet}5$. $2^2{\bullet}3^2$, $3{\bullet}5$. Let p denote the product of each pair. Then p^4 must equal the product of all eight numbers, namely, $2^{12}{\bullet}3^8{\bullet}5^4$. Thus $p = 2^3{\bullet}3^2{\bullet}5 = 360$. Notice that such a pairing is possible: {72, 5}, {8, 45}, {24, 15}, {10, 36}.

Alternate Solution: Since all the numbers are positive, if such a pairing exists then the largest and smallest elements must be paired together: 5•72 = 360.

S10J2 **135**. The incenter lies on the bisectors of the angles of the triangle. Thus $m\angle IAB + m\angle IBA = \frac{1}{2}m\angle CAB + \frac{1}{2}m\angle CBA = \frac{1}{2}\left(m\angle CAB + m\angle CBA\right) = \frac{1}{2}{\bullet}90 = 45$, so $m\angle AIB = 180 - 45 = 135$.

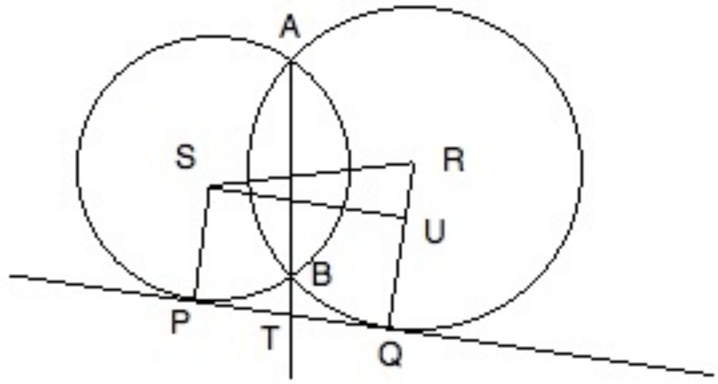
S10J3 **11**. Imagine you are tracing a path from one vertex to the opposite vertex. As you leave each square's interior, you must cross a grid line. Since 5 and 7 are relatively prime, gridlines are crossed one at a time. When you reach the opposite vertex, you will have crossed 7 + 5 = 12 grid lines. However, as you leave the interior of the last square, you cross the last two grid lines simultaneously, so you will have visited the interiors of 11 squares.

S10J4 **3**. Let $x = \sqrt{7+\sqrt{1+\sqrt{7+\sqrt{1+\cdots}}}}$. Then $x = \sqrt{7+\sqrt{1+x}}$. Thus $x^2 - 7 = \sqrt{1+x}$. or in other words $x^4 - 14x^2 - x + 48 = 0$. Since it is given that x is an integer, by the Rational Root Theorem, x is a divisor of 48. Testing the divisors of 48 against the equation $x^2 - 7 = \sqrt{1+x}$. yields 3 as a solution. Because the graphs of $x^2 - 7$ and $\sqrt{1+x}$ intersect at only one point, 3 is the unique solution.

S10J5 **45**. There are nine such subsets whose smallest element is 1, eight whose smallest element is 2, etc. The total number of the desired subsets is therefore $1 + 2 + \cdots + 9 = 45$.

Alternate Solution: Every subset that satisfies the conditions of the problem must have distinct largest and smallest elements. Thus there is a one-to-one correspondence between the desired subsets and pairs of elements of {1, 2, 3, ..., 10}. Therefore the answer is $\binom{10}{2} = (10 * 9)/2 = 45$.

S10J6 $\mathbf{2\sqrt{2}}$. Without loss of generality, assume that B is between T and A. Use Power of a Point to conclude that $TP^2 = TB{\bullet}TA$ and $TQ^2 = TB{\bullet}TA$. Thus $TP = TQ = ½\,PQ$. Let S and R be the centers of the circles with radii 3 and 5, respectively. In quadrilateral $QRSP$, $\overline{SP}$ and $\overline{RQ}$ are perpendicular to $\overline{PQ}$. Draw a line through S perpendicular to $\overline{RQ}$, and let U be their point of intersection. Then $RU = RQ - QU = RQ - SP = 5 - 3 = 2$. Use the Pythagorean Theorem to conclude that $PQ = SU = \sqrt{6^2 - 2^2} = 4\sqrt{2}$, so $PT = 2\sqrt{2}$.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division CONTEST NUMBER 2

Spring 2010 Solutions

S10J7 **200**. There are 25 choices for selecting the first square, then 16 choices for the second selection. Because each two-square selection is counted twice by this method, the number of two-square selections is $(25\bullet 16)/2 = 200$.

S10J8 **11, 12, 13, 14 (ALL REQUIRED)**. Let *S'* be the complement of *S*. Because the sum of the elements of the given set is 120, the sum of the elements of *S'* is 10. Because the number of elements in *S'* can be 1, 2, 3, or 4, (consider for example {10}, {1, 9}, {1, 2, 7} and {1, 2, 3, 4}), the number of elements of *S'* is 15 – *n*, so *n* can be 14, 13, 12, or 11.

S10J9 **9**. The volume of the balloon at 10 seconds is twice its volume at 9 seconds. Because the balloon's volume is 800 at 10 seconds, its volume is 400 at 9 seconds.

S10J10 **(1, 7) and (2, 2) (BOTH REQUIRED)**. The given equation implies that $2 = \sqrt{a + \sqrt{b+2}}$. Thus $4 - a = \sqrt{b+2}$, and so $4 - a \ge 0$, that is, a = 1, 2, 3 or 4. Substitute these values into the last equation to find that the only valid ordered pairs are (1, 7) and (2, 2).

S10J11 **4096**. Because *N* is the fourth power of an integer, the exponent associated with each prime factor of *N* must be a multiple of 4; and because *N* is the sixth power of an integer, the exponent associated with each prime factor of *N* must be a multiple of 6. Thus the exponent associated with each prime factor of *N* must be a multiple of 12, and so *N* is the 12th power of an integer. The smallest value of *N* is therefore $2^{12} = 4096$.

S10J12 $\mathbf{2\sqrt{5}}$. Let *M* and *N* be the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. Then *M* and *N* are the centers of the two circles, so *EM* = *MF* = 3 = *FN* = *NE*. Thus *EMFN* is a rhombus, and its diagonals $\overline{MN}$ and $\overline{EF}$ are perpendicular. Let *O* be their intersection. Apply the Pythagorean Theorem in triangle *MOF* to conclude that $OF = \sqrt{3^2 - 2^2} = \sqrt{5}$. Thus $EF = 2\sqrt{5}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division CONTEST NUMBER 3

Spring 2010 Solutions

S10J13 **3**. The given equation implies that $4^{4x} = \left(2^3\right)^8$. Thus $\left(2^2\right)^{4x} = 2^{24}$, so $2^{8x} = 2^{24}$, and x = 3.

S10J14 **60**. Let the six points be *A*, *B*, *C*, *D*, *E* and *F*. If you trace a path, starting at any vertex, the path can be named by listing its six vertices in the order in which you trace them. Because the vertex at which you start can be any of the six points on the path, and the path can be traced clockwise or counterclockwise, the number of paths is 1/12 the number of permutations of *ABCDEF*, namely, 6!/12 = 60.

S10J15 **0**. Were such an assignment possible, the sum of the numbers preceded by plus signs, including the first one, would equal the sum of the numbers preceded by minus signs. Each of these sums would therefore have to equal half of the sum of the integers from 1 to 10, inclusive, which is 55, that is, each sum would be 55/2, which is impossible.

S10J16 $\mathbf{2 - \sqrt{2}}$. Without loss of generality, assume the larger circle is the one closer to *A*. Let *O* and *P* be the centers of the larger and smaller circles, respectively, and let their radii be $2x$ and x. Draw $\overline{OR}$ perpendicular to $\overline{AB}$ at *R*, and draw $\overline{PS}$ perpendicular to $\overline{BC}$ at S. Then $AR = RO = 2x$, so $AO = 2x\sqrt{2}$, and $PS = SC = x$, so $PC = x\sqrt{2}$. Also, $OP = 2x + x = 3x$. Then $\sqrt{2} = AC = AO + OP + PC = 2x\sqrt{2} + 3x + x\sqrt{2} = 3x + 3x\sqrt{2} = 3x\left(1+\sqrt{2}\right)$, so the sum of the radii equals $3x = \dfrac{\sqrt{2}}{\sqrt{2}+1} = 2 - \sqrt{2}$.

S10J17 **120**. A sequence can have 1, 2, 3 or 4 letters; and there are three choices for each entry in a sequence. Therefore there are $3 + 3^2 + 3^3 + 3^4 = 120$ possible sequences.

S10J18 **36000**. Because n is divisible by 6 and 10, n must be divisible by 30. Thus n has at least one prime factor of 2, 3 and 5. Because the smallest value of n is requested, assume that there is an n with no prime factors other than 2, 3 and 5. Then $n = 2^a \bullet 3^b \bullet 5^c$, where a, b and c are positive integers. Therefore $10n = 2^{a+1} \bullet 3^b \bullet 5^{c+1}$. Because $10n$ is a square, each of these exponents must be even. Hence a is odd, b is even and c is odd. Also, $6n = 2^{a+1} \bullet 3^{b+1} \bullet 5^c$, and because $6n$ is a cube, each of these exponents must be divisible by 3. Thus a and b are each one less than a multiple of 3, and c is a multiple of 3. The smallest permissible values of a, b and c that satisfy these conditions are 5, 2 and 3, respectively, so the minimum value of n is $2^5 \bullet 3^2 \bullet 5^3 = 36000$.