New York City Interscholastic Mathematics League

Senior B Division — Contest Number 1

Part I ***Fall 2009*** ***Contest 1*** ***Time: 10 Minutes***

F09B01 Seven women and five men attended a party. At the party, each man shook hands with every other person once. However, each woman shook hands only with men. How many handshakes took place?

F09B02 Write the following expression in simplest radical form: $\dfrac{\left(\sqrt[3]{12}\right)\left(\sqrt[6]{72}\right)}{\left(\sqrt[3]{\sqrt[4]{108}}\right)\left(\sqrt[3]{\sqrt{3}}\right)}$

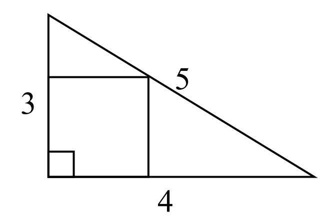
Part II ***Fall 2009*** ***Contest 1*** ***Time: 10 Minutes***

F09B03 A rectangular solid has dimensions 7 inches, 5 inches and 1 inch. If the main diagonal of this solid has the same length as the main diagonal of a cube, find the volume of the cube in cubic inches.

F09B04 The tens digit of a two-digit number is one less than the number's units digit. The product of the number and the number with its digits reversed is 736. Find the original number.

Part III ***Fall 2009*** ***Contest 1*** ***Time: 10 Minutes***

F09B05 A square is inscribed in a 3-4-5 right triangle as shown in the diagram. Find the area of the square.



F09B06 Let A and B be base six digits. Given that the base six division $2A5B_6 \div 15_6$ produces a whole number result, find all possible ordered pairs (A, B).

New York City Interscholastic Mathematics League

Senior B Division Contest Number 2

Part I | **Fall 2009** | **Contest 2** | **Time: 10 Minutes**

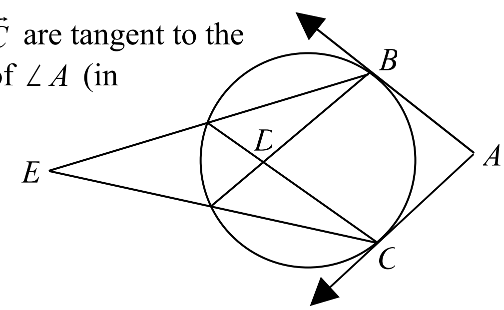
F09B07 In a rectangle whose perimeter is 20, the length of the diagonal is d. How many possible integer values are there for d?

F09B08 Compute the numerical value of $\sum_{n=1}^{100} \cos\left(\frac{n\pi}{6}\right)$ using simplest radical form.

Part II | **Fall 2009** | **Contest 2** | **Time: 10 Minutes**

F09B09 The function $f(x)$ is a linear function that satisfies the equation $f(1+x)-3f(x)=2x$. Find an expression for $f(x)$ for all real values of x.

F09B10 In the diagram, which is not drawn to scale, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are tangent to the circle, $\angle E = 25^\circ$ and $\angle BDC = 75^\circ$. Find the measure of $\angle A$ (in degrees).



Part III | **Fall 2009** | **Contest 2** | **Time: 10 Minutes**

F09B11 For what integer value of k will the equation $3x^3-6x^2-7kx+2k=0$ have one root equal to $2-\sqrt{3}$?

F09B12 The smallest interior angle of a convex polygon with 54 diagonals has a measure of 132°. If the measures of all of the interior angles of the polygon form an arithmetic sequence, find the measure of the largest interior angle of the polygon (in degrees).

New York City Interscholastic Mathematics League

Senior B Division   Contest Number 3

Part I ***Fall 2009*** ***Contest 3*** ***Time: 10 Minutes***

F09B13 Four Chihuahuas and three pups weigh 22 pounds. Three Chihuahuas and two pups weigh 16 pounds. If all of the Chihuahuas have equal weights and all pups have equal weights, find the weight, in pounds, of two Chihuahuas and one pup.

F09B14 How many integers satisfy the following system of inequalities?

$$|4x+8| < 33 \text{ and } |3x-6| \geq 8$$

Part II ***Fall 2009*** ***Contest 3*** ***Time: 10 Minutes***

F09B15 Rectangle A has 20% more area than rectangle B. Rectangle C has 15% less area than rectangle A. Rectangle C has X% more area than rectangle B. Compute the value of X.

F09B16 Given the five numbers 17, 4, 28, 23 and x, find all positive values of x such that the median of the five numbers is equal to their arithmetic mean.

Part III ***Fall 2009*** ***Contest 3*** ***Time: 10 Minutes***

F09B17 Find the two smallest positive degree measures for x satisfying the equation $\sin x = \cos^2(14.5^\circ) - \sin^2(14.5^\circ)$.

F09B18 For any integer $n \geq 1$, $f(2n) = f(n) + 1$ and $f(2n+1) = f(n) - 1$. If $f(1) = 0$, compute the value of $f(2009)$.

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Senior B Division   Contest Number 4

Part I   ***Fall 2009***   ***Contest 4***   ***Time: 10 Minutes***

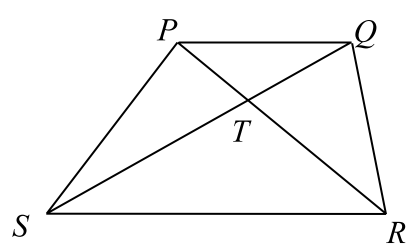
F09B19   Let x be an integer and D be the thousands digit in the base-ten number $4D631$. Find the value of x that satisfies the equation $9x + 18 = 4D631$.

F09B20   Solve the following equation for x: $\log_2 x + \log_2(x - 1) = \log_2(2x - 3) + 1$.

Part II   ***Fall 2009***   ***Contest 4***   ***Time: 10 Minutes***

F09B21   If $x + y = 1$ and $x^2 + y^2 = 2$, find the numerical value of $x^3 + y^3$.

F09B22   The diagram below, which is not drawn to scale, shows trapezoid $PQRS$ with bases $\overline{PQ}$ and $\overline{RS}$. The diagonals of the trapezoid intersect at point T. If $\frac{PT}{TR} = \frac{3}{7}$, compute the value of the fraction $\frac{[\Delta PQT]}{[PQRS]}$, where the brackets represent area.



Part III   ***Fall 2009***   ***Contest 4***   ***Time: 10 Minutes***

F09B23   Find the complex number c such that the equation $x^2 + 4x + 6ix + c = 0$ has only one solution. Express your answer in the form $a + bi$, where $i = \sqrt{-1}$.

F09B24   There are only two lines containing the point $(9, -1)$ that form a right triangle with the x-axis and y-axis having an area of 6 square units. Compute the slopes of these two lines.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 5

PART I ***FALL 2009*** ***CONTEST 5*** ***TIME: 10 MINUTES***

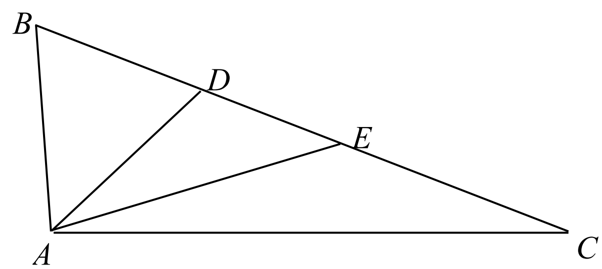
F09B25 Find the smallest positive four-digit number divisible by 8, 15, 18 and 24.

F09B26 If you have fifty coins consisting of nickels, dimes, and quarters worth exactly four dollars and 65 cents, what is the maximum number of quarters that you can have?

PART II ***FALL 2009*** ***CONTEST 5*** ***TIME: 10 MINUTES***

F09B27 When the expression $(2x+y)^4$ is multiplied out, what is the sum of its coefficients?

F09B28 In the diagram below, $\overline{AD}$ bisects $\angle BAC$, $\overline{AE}$ bisects $\angle CAD$, and $\angle ABC : \angle AEB : \angle ACB = 6:3:1$. Find the measure of $\angle AEC$ in degrees.



PART III ***FALL 2009*** ***CONTEST 5*** ***TIME: 10 MINUTES***

F09B29 Compute all real values of x that satisfy the equation $6x^2+12=17|x|$.

F09B30 Bag A contains 6 blue and 4 yellow marbles. Bag B contains 7 blue and 3 yellow marbles. If 2 marbles are drawn at random from each bag, without replacement, compute the probability that from the four marbles drawn, there will be 2 blue and 2 yellow marbles.

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Senior B Division Contest Number 1

Fall 2009 Solutions

F09B01. **45**. Each man shook hands with 7 women for a total of $5 \cdot 7 = 35$ handshakes between men and women. Now, each man shook the hands of four other men, but this counts each man-to-man handshake twice. So, there were a total of $\frac{5 \cdot 4}{2} = 10$ handshakes between men. Therefore, there were a total of $35 + 10 = 45$ handshakes. Alternately, each of the 5 men shook hands with 11 other people and each of the 7 women shook hands with 5 other people. Since this counts each handshake twice (once for each of the two people shaking hands), the total number of handshakes is $(5 \bullet 11 + 7 \bullet 5)/2 = 45$.

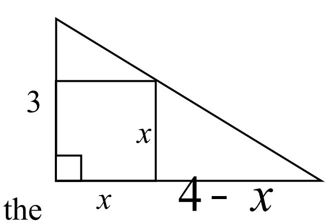
F09B02. $\mathbf{2\sqrt[4]{3}}$. Simplify by factoring and using the properties of rational exponents:

$$\frac{\left(\sqrt[3]{12}\right)\left(\sqrt[6]{72}\right)}{\left(\sqrt[3]{\sqrt[4]{108}}\right)\left(\sqrt[3]{\sqrt{3}}\right)} = \frac{\sqrt[3]{2^2 \cdot 3} \cdot \sqrt[6]{2^3 \cdot 3^2}}{\sqrt[12]{2^2 \cdot 3^3} \cdot \sqrt[6]{3}} = \frac{2^{2/3} \cdot 3^{1/3} \cdot 2^{1/2} \cdot 3^{1/3}}{2^{1/6} \cdot 3^{1/4} \cdot 3^{1/6}} = 2^{2/3 + 1/2 - 1/6} \cdot 3^{1/3 + 1/3 - 1/4 - 1/6} = 2 \cdot 3^{1/4} = 2\sqrt[4]{3}.$$

F09B03. **125**. Given the length, *l*, width, *w*, and height, *h*, of a rectangular solid, the length of the solid's main diagonal, *d*, is $d = \sqrt{l^2 + w^2 + h^2}$. As a result of this formula, the main diagonal of a cube is $e\sqrt{3}$, where *e* is the length of one of the cube's edges. For this problem, the length of the rectangular solid's diagonal is $\sqrt{7^2 + 5^2 + 1^2} = \sqrt{49 + 25 + 1} = \sqrt{75} = 5\sqrt{3}$. Thus the length of the edge of the cube is 5, and therefore the volume of the cube is $5^3 = 125$.

F09B04. **23**. Let T = the tens digit and U = the units digit. We have two equations with the 2 unknowns: $T = U - 1$ and $(10T + U)(10U + T) = 736$. We can find the value of U by using substitution and solving the resulting quadratic equation. We have $(10(U - 1) + U)(10U + U - 1) = 736$ $\Rightarrow (11U - 10)(11U - 1) = 736 \Rightarrow 121U^2 - 121U - 726 = 0 \Rightarrow 121(U^2 - U - 6) = 0 \Rightarrow (U - 3)(U + 2) = 0$. Therefore, U = 3 and the original number is 23. (To check, multiply 23 and 32). Another way to solve this problem is to realize that 34*43 > 30*40 = 1200, which is too large, and 12*21 < 20*30 = 600, which is too small, so the answer must be 23.

F09B05. $\mathbf{\frac{144}{49} = 2\frac{46}{49}}$. Let the length of a side of the square = x. The two triangles inside of the larger 3-4-5 triangle are similar to each other, in addition to being similar to the 3-4-5 triangle. Find a suitable correspondence, set up a proportion, and then solve for x^2. Here's one way using the smaller triangle to the lower right and the 3-4-5 triangle: $\frac{x}{3} = \frac{4 - x}{4} \Rightarrow 4x = 12 - 3x \Rightarrow x = \frac{12}{7} \Rightarrow x^2 = \frac{144}{49}$.



F09B06. **(0, 0), (2, 5) and (3, 2).** Using the definition for base six, $2A5B_6 \div 15_6 = (2 \cdot 6^3 + A \cdot 6^2 + 5 \cdot 6 + B) \div (1 \cdot 6 + 5) = (462 + 36A + B) \div 11$. Since 462 is divisible by 11, if this division is to be a whole number, $36A + B$ must be divisible by 11. Dividing $36A + B$ by 11 leaves a remainder of $3A + B$, so we need $3A + B$ to be divisible by 11. Since A and B are base six digits, A and B are the integers from 0 to 5. If A = 0, then B = 0. If A = 1, then B = 8, but this is impossible. If A = 2, then B = 5. If A = 3, then B = 2. If A = 4, then B = 10; this is impossible. Finally, if A = 5, B = 7; this is impossible. Therefore the answers are (0, 0), (2, 5) and (3, 2).

New York City Interscholastic Mathematics League

Senior B Division Contest Number 2

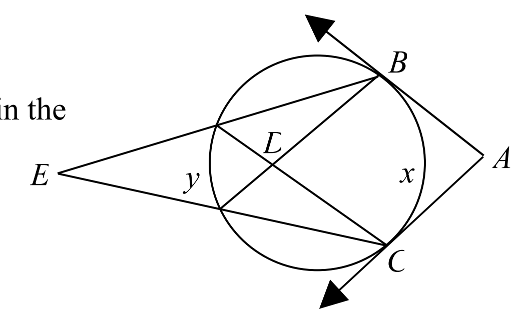
Fall 2009 Solutions

F09B07. **2**. If we allow the “shape” of the rectangle to vary we would observe two extremes. First, the rectangle could be a square with sides of length 5, and as a result the length of the diagonal would be $5\sqrt{2}$. On the other hand, as one dimension of the rectangle shrinks, the other dimension must increase in order to maintain the condition that the perimeter is 20. The second extreme is a rectangle with one dimension very close to 10 and the other dimension very close to zero. The diagonal of this rectangle would have a length very close to 10. Therefore, the set of all possible diagonal lengths d is $5\sqrt{2} \le d < 10$. Since $5\sqrt{2} \approx 7.1$, the only possible integer lengths for the diagonal are 8 and 9.

F09B08. $\mathbf{\dfrac{\sqrt{3}}{2}}$. Beginning at zero, in the first period for cosine, the sum of the cosines for the 12 angles equals zero because cosine is positive in two quadrants and negative in the other two quadrants. That is, $\sum_{n=1}^{12} \cos\left(\frac{n\pi}{6}\right) = 0$. As a result, the sum will be zero over any number of complete periods of cosine. So, $\sum_{n=1}^{96} \cos\left(\frac{n\pi}{6}\right) = 0$ and as a result, $\sum_{n=1}^{100} \cos\left(\frac{n\pi}{6}\right) = \sum_{n=1}^{4} \cos\left(\frac{n\pi}{6}\right)$

$$= \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2} + 0 - \frac{1}{2} = \frac{\sqrt{3}}{2}.$$

F09B09. $\mathbf{-x - \frac{1}{2} = -\left(x + \frac{1}{2}\right)}$. Let $f(x) = ax + b$. Then, $a(1+x) + b - 3(ax+b) = 2x$ $\Rightarrow a + ax + b - 3ax - 3b = 2x \Rightarrow -2ax + (a - 2b) = 2x$. So, $-2a = 2 \Rightarrow a = -1$ and as a result $-1 - 2b = 0 \Rightarrow b = -\frac{1}{2}$. Hence, $f(x) = -x - \frac{1}{2}$.



F09B10. **80°**. Let x and y be the measure of the arcs as indicated in the diagram. Using the appropriate relationships between arcs and angles in a circle, we have $\frac{x-y}{2} = 25$ and $\frac{x+y}{2} = 75$. Add these two equations to obtain $x = 100$. Therefore, $\angle A = 180 - 100 = 80$.

F09B11. **3**. Divide both sides of the equation by 3 to obtain $x^3 - 2x^2 - \frac{7k}{3}x + \frac{2k}{3} = 0$. From this equation we can see that the sum of the roots is 2 and the product of the roots is $-\frac{2k}{3}$. Since one of the roots is $2 - \sqrt{3}$, another other root must be $2 + \sqrt{3}$, since irrational roots of polynomials with rational coefficients are found in conjugate pairs. Let r = the third root. Then the sum of the roots equals 2 $\Rightarrow (2 - \sqrt{3}) + (2 + \sqrt{3}) + r = 2 \Rightarrow r = -2$. Next, k can be found by using product of all three roots. The product of all three roots is $-2(2 - \sqrt{3})(2 + \sqrt{3}) = -2$ and $\frac{-2k}{3} = -2 \Rightarrow k = 3$.

F09B12. **168°**. It is convenient to know three things: 1) the number of diagonals in an *n*-gon is ${}_nC_2 - n = \dfrac{n(n-1)}{2} - n = \dfrac{n(n-3)}{2}$, 2) the sum of the interior angles of an n-gon is $(n-2)\cdot 180^\circ$, and 3) the sum of an arithmetic series with *n* terms, given the first term a_1 and the last term a_n, is $\dfrac{n(a_1+a_n)}{2}$. For this problem we have $\dfrac{n(n-3)}{2} = 54 \Rightarrow n^2 - 3n - 108 = 0 \Rightarrow (n-12)(n+9) = 0 \Rightarrow n = 12$. So, the polygon has 12 sides and the sum of the interior angles is $10\cdot 180^\circ = 1800^\circ$. Now, let *A* be the measure of the largest angle of the polygon. Since the angles of the polygon form an arithmetic sequence beginning with 132 and whose sum is 1800, we have $\dfrac{12(132+A)}{2} = 1800 \Rightarrow 132 + A = 300 \Rightarrow A = 168$.

New York City Interscholastic Mathematics League

Senior B Division Contest Number 3

Fall 2009 Solutions

F09B13. **10**. One way to solve the problem is to solve a pair of linear equations simultaneously. Let C = the weight of one Chihuahua (in pounds) and let P = the weight of one pup (also in pounds). Then the first sentence translates as $4C+3P=22$ and the second sentence becomes $3C+2P=16$. Then, using linear combinations, multiply the first equation by 2 and multiply the second equation by 3, yielding the equations $8C+6P=44$ and $9C+6P=48$. Now, subtracting the first of these equations from the second gives $C=4$. Substituting this value into either of the original equations gives $P=2$. Finally, this means that two Chihuahuas and one pup weigh $2(4)+2=10$ pounds. We can even solve this problem without solving for the values of C and P. If we subtract $3C+2P=16$ from $4C+3P=22$, we get $C+P=6$, and subtracting that from $3C+2P=16$ gives us $2C+P=10$.

F09B14. **12**. For the first inequality, we have: $-33<4x+8<33 \Rightarrow -41<4x<25$ $\Rightarrow -10\frac{1}{4}<x<6\frac{1}{4}$. For the second inequality, we have $3x-6\geq 8$ or $3x-6\leq -8$. Solving the first case yields $3x-6\geq 8 \Rightarrow 3x\geq 14 \Rightarrow x\geq 4\frac{2}{3}$, and in the second case, $3x-6\leq -8 \Rightarrow 3x\leq -2 \Rightarrow x\leq -\frac{2}{3}$. As a result, the intersection of both solution sets is $-10\frac{1}{4}<x\leq -\frac{2}{3}$ and $4\frac{2}{3}\leq x<6\frac{1}{4}$. The integers in the solution set are -10, -9,…, -2, -1, 5 and 6. Therefore, there are 12 integer solutions.

F09B15. **2**. Let A, B, and C = the areas of rectangles A, B and C, respectively. Then, from the first piece of information, $A=1.2B$, and from the second piece of information, $C=0.85A$. Substituting $A=1.2B$ into the second equation gives $C=0.85(1.2B)=1.02B$. Therefore, rectangle C has 2% more area than rectangle B. (Notice that the last bit of arithmetic is easier with fractions! That is, $C=\frac{17}{20}\left(\frac{6}{5}B\right)=\frac{102}{100}B$.)

F09B16. **13, 18, 43**. The mean of 4, 17, 23, 28 and x is $\frac{4+17+23+28+x}{5}=\frac{72+x}{5}$. If $x\leq 17$, then $\frac{72+x}{5}=17 \Rightarrow x=13$. If $17<x\leq 23$, then $\frac{72+x}{5}=x \Rightarrow x=18$. Finally, if $x>23$, then $\frac{72+x}{5}=23 \Rightarrow x=43$. The answer is 13, 18, or 43.

F09B17. **61°,119°**. Three trigonometric ideas are important for this problem: the double angle formula for cosine, $\cos 2x=\cos^2 x-\sin^2 x$; and the sine/cosine relationship, $\sin(90^\circ-x)=\cos x$, and the fact that sine is positive in the second quadrant. By the double angle formula for cosine, the equation simplifies to $\sin x=\cos 29^\circ$. Then, by the sine/cosine relationship, $x=90^\circ-29^\circ=61^\circ$. This is the first answer. Since sine is positive in the second quadrant and using 61° as a reference angle, the second answer is $180^\circ-61^\circ=119^\circ$.

F09B18 **-4**. Here's the long way: Using the first definition when the argument of the function is even and using the second definition when the function is odd gives the following chain of equivalencies: $f(2009)=f(2\cdot 1004+1)=f(1004)-1=f(2\cdot 502)-1=f(502)+1-1$
$=f(2\cdot 251)=f(251)+1=f(2\cdot 125+1)+1=f(125)-1+1=f(2\cdot 62+1)=f(62)-1$
$=f(2\cdot 31)-1=f(31)+1-1=f(2\cdot 15+1)=f(15)-1=f(2\cdot 7+1)-1=f(7)-1-1$

$= f(2 \cdot 3 + 1) - 2 = f(3) - 1 - 2 = f(2 \cdot 1 + 1) - 3 = f(1) - 1 - 3 = -4$. It is easier to realize that upon successively dividing the argument of the function by 2 and truncating at the decimal point, the even arguments provide +1 for the function and odd arguments provide -1 for the function. For $f(2009)$ we have the sequence 2009, 1004, 502, 251, 125, 62, 31, 15, 7, 3, 1. Therefore, we have $f(2009) = -1 + 1 + 1 - 1 - 1 + 1 - 1 - 1 - 1 - 1 + 0 = -4$.

As an extension of this problem, it can be proven by induction on the number of base 2 digits that for every positive integer n, *f*(n) = 1 + number of zeroes in the base 2 representation of n – number of ones in the base 2 representation of n. (Going from n to 2n is equivalent to adding a 0 on the right end of a binary number, while going from n to 2n + 1 is equivalent to adding a 1.) For example, the base 2 representation of 2009 is 11111011001, so *f*(2009) = 1 + 3 – 8 = -4.

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Senior B Division Contest Number 4

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F09B19. **4957**. Subtract 18 from both sides of the equation to obtain $9x = 4D613$. If x is to be an integer, 4D613 must be divisible by 9. This means that the sum of the digits of 4D613 must be divisible by 9. We have $4 + D + 6 + 1 + 3 = 14 + D \Rightarrow D = 4$. Finally, if $9x = 44613$, then $x = 4957$.

F09B20. **2, 3**. Using the sum to product property for logarithms, both sides of the equation can be written as a logarithm with base 2: $\log_2 x + \log_2(x - 1) = \log_2(2x - 3) + 1$
$\Rightarrow \log_2 x + \log_2(x - 1) = \log_2(2x - 3) + \log_2 2 \Rightarrow \log_2 x(x - 1) = \log_2 2(2x + 3)$. Then, setting the arguments equal, we have: $x(x - 1) = 2(2x - 3) \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow (x - 2)(x - 3) = 0$. Therefore, the answers are 2 and 3. (Upon solving logarithmic equations, it is always wise to check to make sure neither of the answers are extraneous, and in this case, they aren't!)

F09B21. $\mathbf{\frac{5}{2} = 2\frac{1}{2} = 2.5}$. Using the "sum of two cubes formula," we have $x^3 + y^3$
$= (x + y)(x^2 - xy + y^2)$. Substituting 1 for $x + y$ and 2 for $x^2 + y^2$ yields $x^3 + y^3 = 1 \cdot (2 - xy)$. To find the value of xy we can expand $(x + y)^2$, set the result equal to 1 and solve for xy. This gives us
$(x + y)^2 = x^2 + y^2 + 2xy = 1 \Rightarrow xy = -\frac{1}{2}$. Therefore, $x^3 + y^3 = \frac{5}{2}$.

F09B22. $\mathbf{\frac{9}{100} = 0.09}$. Because the bases of a trapezoid are parallel, $\angle PQS = \angle RSQ$ and $\angle QPR = \angle SRQ$ (given parallel lines and a transversal, alternate interior angles are equal). Therefore, $\Delta PQT : \Delta RST$ by the Angle-Angle Similarity Theorem. As a result, $\frac{PT}{TR} = \frac{QT}{TS} = \frac{3}{7}$. Since, $\frac{PT}{TR} = \frac{3}{7}$, the ratio of $[\Delta PQT]$ to $[\Delta RST]$ is $\left(\frac{3}{7}\right)^2 = \frac{9}{49}$. Also, the ratio of $[\Delta PQT]$ to $[\Delta QTR]$ is $\frac{3}{7}$ because both triangles have the same altitude corresponding to bases having the ratio $\frac{3}{7}$. In addition, since $\frac{QT}{TS} = \frac{3}{7}$, the ratio of $[\Delta PQT]$ to $[\Delta PTS]$ is $\frac{3}{7}$. Now, let $[\Delta PQT] = 9x$. Then $[\Delta RST] = 49x$, $[\Delta PTS] = 21x$, and $[\Delta QTR] = 21x$. Therefore $[PQRS] = 9x + 49x + 21x + 21x = 100x$. Hence, $\frac{[\Delta PQT]}{[PQRS]} = \frac{9x}{100x} = \frac{9}{100}$.

F09B23. $\mathbf{-5 + 12i}$. The quadratic equation will have one solution when the discriminant equals zero. For this equation, the discriminant equals $(4 + 6i)^2 - 4c$. Set the discriminant equal to zero and solve for c: $(4 + 6i)^2 - 4c = 0 \Rightarrow 16 + 48i - 36 - 4c = 0 \Rightarrow -4c = 20 - 48i \Rightarrow c = -5 + 12i$.

F09B24. $\mathbf{-\frac{1}{3}, -\frac{1}{27}}$. Let the slope of the unknown lines = m. Using point-slope form the equation for the lines must have the form $y + 1 = m(x - 9)$. Letting x = 0, we know that the y-intercept of this line is $y = -9m - 1$. Letting y = 0 gives the x-intercept as $x = \frac{1}{m} + 9 = \frac{9m+1}{m}$. Hence the area of the triangle formed by the line and the axes is $\frac{1}{2}\left|(-9m-1)\left(\frac{9m+1}{m}\right)\right| = \frac{1}{2} \bullet \frac{(9m+1)^2}{|m|} = 6$. This is equivalent to $(9m+1)^2 = 12m$ when m > 0 and $(9m+1)^2 = -12m$ when m < 0. In the first case, the equation simplifies to $81m^2 + 6m + 1 = 0$, whose roots are imaginary. In the second case, the equation simplifies to $81m^2 + 30m + 1 = 0$. We can factor this equation into (27m + 1)(3m + 1) = 0, so therefore m = -1/27 or m = -1/3.

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Senior B Division Contest Number 5

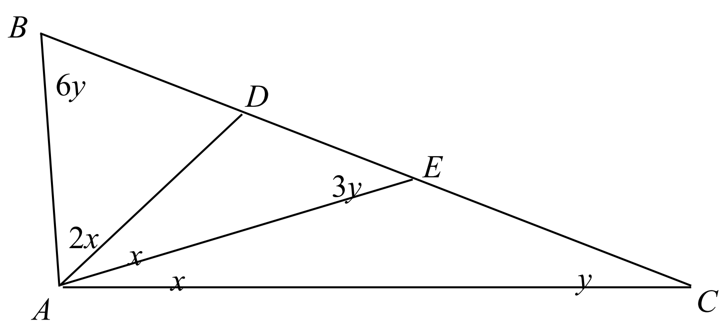
Fall 2009 Solutions

F09B25. **1080**. The prime factorizations are $8=2^3$, $15=3\cdot 5$, $18=2\cdot 3^2$ and $24=2^3\cdot 3$. Therefore the least common multiple of these four numbers is $2^3\cdot 3^2\cdot 5=360$. Since $360\cdot 2=720$ and $360\cdot 3=1{,}080$, the smallest positive four-digit number is 1,080.

F09B26. **10**. Let N = the number of nickels, D = the number of dimes and Q = the number of quarters. Then, from the given information, we have $N+D+Q=50$ (call this equation #1) and $5N+10D+25Q=465 \Rightarrow N+2D+5Q=93$ (call this equation #2). Subtracting equation #1 from equation #2 yields $D+4Q=43$. We can maximize the number of quarters used by letting $D=3$ and $Q=10$. Therefore the maximum number of quarters is 10. The final configuration of coins is 10 quarters, 3 dimes, and 37 nickels.

F09B27. **81**. Expanding yields $(2x+y)^4=16x^4+32x^3y+24x^2y^2+8xy^3+y^4$. Hence the sum of the coefficients is $16+32+24+8+1=81$. Alternatively, the sum of the coefficients of a polynomial is the value of the polynomial when each of the variables has a value of 1. In this case, with $x=y=1$, the sum of the coefficients is $(2+1)^4=81$.

F09B28. $\mathbf{144^\circ}$. From the given information we can mark the diagram as shown below. Since the angle marked $3y$ is exterior to triangle AEC, the Exterior Angle Theorem yields $3y=x+y \Rightarrow x=2y$. Also, using the fact that the angles of a triangle sum to 180 degrees, in triangle ABC we have $4x+7y=180^\circ$. Substituting $x=2y$ into this last equation yields $15y=180^\circ \Rightarrow y=12^\circ$. Finally, since $\angle AEC$ and the angle marked $3y$ form a linear pair, we have $\angle AEC=180^\circ-3(12^\circ)=144^\circ$.



F09B29. $\mathbf{\pm\frac{3}{2},\pm\frac{4}{3}}$. Since $x^2=|x|^2$ for all real values of x, we have $6x^2+12=17|x| \Rightarrow 6|x|^2-17|x|+12=0$. This equation is quadratic in $|x|$ and the left-hand side can be factored: $6|x|^2-17|x|+12=0 \Rightarrow (2|x|-3)(3|x|-4)=0 \Rightarrow 2|x|=3$ and $3|x|=4$. Therefore, $x=\pm\frac{3}{2}$ or $x=\pm\frac{4}{3}$.

F09B30. $\mathbf{\frac{1}{3}}$. There are three possibilities: 1) one blue and one yellow marble are selected from each bag, 2) two blue marbles are selected from bag A and 2 yellow marbles are selected from bag B, and 3) two yellow marbles are selected from bag A and two blue marbles are selected from bag B. The probability that 1) occurs is $\frac{{}_6C_1 \cdot {}_4C_1}{{}_{10}C_2} \times \frac{{}_7C_1 \cdot {}_3C_1}{{}_{10}C_2} = \frac{24 \cdot 21}{45 \cdot 45} = \frac{504}{45 \cdot 45}$. (Note that without a calculator, it is best to leave these fractions conveniently unsimplified.) The probability that 2) occurs is $\frac{{}_6C_2}{{}_{10}C_2} \times \frac{{}_3C_2}{{}_{10}C_2} = \frac{15 \cdot 3}{45 \cdot 45} = \frac{45}{45 \cdot 45}$. The probability that 3) occurs is $\frac{{}_4C_2}{{}_{10}C_2} \times \frac{{}_7C_2}{{}_{10}C_2} = \frac{6 \cdot 21}{45 \cdot 45} = \frac{126}{45 \cdot 45}$. Hence, the probability that two of the marbles are blue and two are yellow is the sum of the above three probabilities: $\frac{504}{45 \cdot 45} + \frac{45}{45 \cdot 45} + \frac{126}{45 \cdot 45} = \frac{675}{45 \cdot 45} = \frac{1}{3}$.

Another way, which involves fewer large numbers, is to break down the probabilities by bag. For bag A, the probability of drawing two blue marbles is $\left(\frac{6}{10}\right)\left(\frac{5}{9}\right) = \frac{30}{90} = \frac{5}{15}$, the probability of drawing two yellow marbles is $\left(\frac{4}{10}\right)\left(\frac{3}{9}\right) = \frac{12}{90} = \frac{2}{15}$, and the probability of drawing one blue and one yellow marble must be $\frac{8}{15}$ for the probabilities to add up to 1. For bag B, the probability of drawing two blue marbles is $\left(\frac{7}{10}\right)\left(\frac{6}{9}\right) = \frac{42}{90} = \frac{7}{15}$, the probability of drawing two yellow marbles is $\left(\frac{3}{10}\right)\left(\frac{2}{9}\right) = \frac{6}{90} = \frac{1}{15}$, and the probability of drawing one blue and one yellow marble must be $\frac{7}{15}$ for the probabilities to add up to 1. Now we can use these values to show that the probability of 1) is $\left(\frac{7}{15}\right)\left(\frac{8}{15}\right) = \frac{56}{225}$, the probability of 2) is $\left(\frac{5}{15}\right)\left(\frac{1}{15}\right) = \frac{5}{225}$, and the probability of 3) is $\left(\frac{2}{15}\right)\left(\frac{7}{15}\right) = \frac{14}{225}$. The sum of the three cases is $\frac{56}{225} + \frac{5}{225} + \frac{14}{225} = \frac{75}{225} = \frac{1}{3}$.