New York City Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 1

PART I *FALL 2007* *CONTEST 1* *TIME: 10 MINUTES*

F07A1 Compute the number of possible ways to place 6 identical stones on a 6×6 chessboard so that no two lie in the same row or same column.

F07A2 Regular pentagon *ABCDE* has sides of length 3. Let *R* be the region composed of points outside *ABCDE* which are at a distance of at most 1 from some point of *ABCDE*. Compute the area of *R*.

PART II *FALL 2007* *CONTEST 1* *TIME: 10 MINUTES*

F07A3 Let $x=\log_2 3^2\cdot\log_3 4^3\cdot\log_4 5^4\cdot\log_5 6^5\cdot\log_6 7^6\cdot\log_7 8^7$. Compute x.

F07A4 Euler drove from Basel to St. Petersburg. For half the distance of his trip he traveled at 60 miles per hour (mph). For half the time of his trip he traveled at 40 mph. For the remaining portion of his trip he traveled at 50 mph. Compute his average speed for the whole trip, in mph.

PART III *FALL 2007* *CONTEST 1* *TIME: 10 MINUTES*

F07A5 A non-constant geometric progression of real numbers has the property that its 1st, 3rd and 5th terms form an arithmetic progression. Compute the common ratio of the geometric progression.

F07A6 Non-degenerate triangle *ABC* has vertices $A(-1,0)$, $B(1,0)$ and $C(s,t)$ and the lines $x=1$ and $y=2$ each divide the triangle into two regions of equal area. Compute the ordered pair (s,t).

ANSWERS:		
	F07A1	**720**
	F07A2	$15+\pi$
	F07A3	**15120**
	F07A4	**540/11**
	F07A5	**-1**
	F07A6	$\left(3,4+2\sqrt{2}\right)$

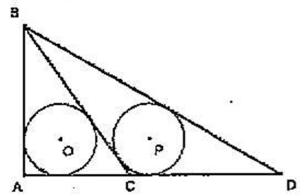
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division **CONTEST NUMBER 2**

PART I ***FALL 2007*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F07A7 If $\cos t = \frac{2}{\sqrt{5}}\left(0^\circ < t < 90^\circ\right)$, compute the value of $\left(\log_5 \cos t\right) + \left(\log_5 \tan t\right)$.

F07A8 In ΔABC with right angle A as in the diagram below, $AB = 4$ and $AC = 3$. Circle O is inscribed in ΔABC. $\overline{AC}$ is extended to point D such that circle P is inscribed in ΔBCD. If both circles have the same radius, compute CD.



PART II ***FALL 2007*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F07A9 Two right circular cones of height 12 and base radius 4 are situated in space so that the vertex of each is the center of the base of the other. Compute the volume of their intersection.

F07A10 If $f(0) = 3$ and for each integer $n \geq 0$, $f(n+1) = 3 \cdot f(0) \cdot f(1) \cdot f(2) \cdot \cdots \cdot f(n)$, then $f(10) = p^a$, for some prime p and integer a, compute (p, a).

PART III ***FALL 2007*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F07A11 If $x = \sqrt{2+\sqrt{6+\sqrt{2+\sqrt{6+\ldots}}}}$ is a root of $x^4 + hx^2 + ax + m = 0$, compute the ordered triple (h, a, m).

F07A12 A jar holds g green and 17 blue marbles, for integer g. Pascal reaches into the jar and selects three marbles at random, without replacement. Compute all three possible values of g so that the probability that Pascal chooses three green marbles is the same as the probability that Pascal chooses two green and one blue marble.

ANSWERS:				
	F07A7	**-1/2**	F07A10	**(3, 1024)**
	F07A8	**9/2**	F07A11	**(-4, -1, -2)**
	F07A9	**16π**	F07A12	**0, 1, 53**

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NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 3

PART I *FALL 2007* *CONTEST 3* *TIME: 10 MINUTES*

F07A13 Compute n such that a regular n-gon has twice as many diagonals as sides.

F07A14 There are four given numbers, w, x, y and z, and there are six pairwise sums which can be formed, $w+x, w+y, w+z, x+y, x+z$ and $y+z$. For some choice of w, x, y and z, five of these sums are 13, 25, 36, 39 and 48. Compute the sixth sum.

PART II *FALL 2007* *CONTEST 3* *TIME: 10 MINUTES*

F07A15 Let R be the portion of the interior of unit square $ABCD$ composed of all points X such that: $AX < BX < DX < CX$. Compute the area of R.

F07A16 A positive integer n is called a *b-d-palindrome* if its representation in base b is the same as its representation in base d except with the digits reversed (no leading 0's allowed). For example, 51 is a 5-7-palindrome because $201_5 = 51 = 102_7$. Compute both two-digit 7-10-palindromes. (Express your answer in base 10.)

PART III *FALL 2007* *CONTEST 3* *TIME: 10 MINUTES*

F07A17 Compute the value of the expression:

$$6\sin^2 15^\circ \cos^2 15^\circ + 5\cos^4 15^\circ + 4\sin^2 15^\circ + \sin^4 15^\circ.$$

F07A18 Archimedes has a machine which outputs integers between 1 and 10, inclusive, so that each integer has a fixed probability (possibly 0) of being produced. The expected value of each output is 7 and the probability that an output is equal to 10 is p. Compute the largest possible value of p.

ANSWERS:

F07A13	**7**
F07A14	**22**
F07A15	**1/8**
F07A16	**23, 46**
F07A17	**5**
F07A18	**2/3**

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NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 4

PART I *FALL 2007* *CONTEST 4* *TIME: 10 MINUTES*

F07A19 Each letter in the word BERNOULLI is written on a scrap of paper (so, there are 9 scraps of paper in use). Four of these scraps are selected at random to make a new "word." Compute the probability that this new word contains exactly one "L".

F07A20 Compute the prime number p such that 16! ends in the same number of zeroes when it is written in base 16 as it does when it is written in base p.

PART II *FALL 2007* *CONTEST 4* *TIME: 10 MINUTES*

F07A21 The sequence $a_1, a_2, a_3, \ldots$ is a non-constant arithmetic progression and the terms a_1, a_2 and a_4 form a geometric progression. Compute the ratio: $\frac{a_6}{a_5}$.

F07A22 If $x = \sqrt[6]{3}$, compute the value of the expression:
$(x+1)(x^2+1)(x^2+x+1)(x-1)(x^4-x^2+1)(x^2-x+1)$.

PART III *FALL 2007* *CONTEST 4* *TIME: 10 MINUTES*

F07A23 S is a set with 6 elements. Gauss chooses subsets of S with 3 elements so that no two of the chosen subsets have more than 1 element in common. Compute the greatest number of such subsets that Gauss can choose.

F07A24 Points A, B, C and D are given in the plane so that ΔABC is equilateral with side length 3 and D is equidistant from A, B and C. Three circles are drawn so that each passes through D and two of A, B, C. Compute the total area enclosed by these circles.

ANSWERS:

F07A19	**5/9**
F07A20	**5**
F07A21	**6/5**
F07A22	**8**
F07A23	**4**
F07A24	$6\pi + \frac{9\sqrt{3}}{2}$ or $\frac{12\pi + 9\sqrt{3}}{2}$

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NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 5

PART I *FALL 2007* *CONTEST 5* *TIME: 10 MINUTES*

F07A25 iTems come in packages of three and five. Cauchy has a coupon so that each package of three iTems costs $2 while each package of five iTems costs $4. Compute the smallest amount of money, in dollars, that Cauchy can pay to purchase exactly 35 iTems.

F07A26 *ABCDEF* is a hexagon, all of whose interior angles measure 120 degrees, $AB = CD = 3$, $EF = 1$ and $BC = 6$. Compute *DE*.

PART II *FALL 2007* *CONTEST 5* *TIME: 10 MINUTES*

F07A27 If *x* and *y* are two positive real numbers such that $6x^2 - 13xy + 5y^2 = 0$, compute all possible values of $\frac{x^2 - y^2}{x^2 + y^2}$.

F07A28 Compute the smallest positive integer *n* such that $25n$ leaves a remainder of 1 when divided by 49 and $49n$ leaves a remainder of 1 when divided by 25.

PART III *FALL 2007* *CONTEST 5* *TIME: 10 MINUTES*

F07A29 Two sequences are defined recursively by the equations $a_1 = 3$, $b_1 = 2$, $a_{n+1} = a_n + b_n$ and $b_{n+1} = a_n - b_n$ for $n \geq 1$. Compute a_{13}.

F07A30 A rotation in the plane by an angle θ centered at the point (a, b) maps (0, 5) to $(-8/5, 29/5)$ and maps (2, -9) to $(26/5, -33/5)$. Compute the ordered pair (a, b).

ANSWERS:

F07A25	**24**
F07A26	**8**
F07A27	$\frac{-3}{5}, \frac{8}{17}$
F07A28	**149**
F07A29	**192**
F07A30	**(-3, 1)**

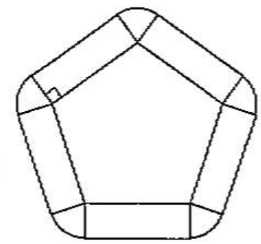
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 1

Fall 2007 Solutions

F07A1 **720**. Method 1: There must be one stone in each row. In the first row, the stone has 6 different columns into which it may be placed. Then in the second row, there are 5 remaining columns, in the third row there are 4, in the fourth row 3, in the fifth row 2 and in the sixth row only 1 remaining column. Thus the answer is $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

Method 2: There are 36 possible places for the first stone. This leaves 25 squares for the second, 16 for the third, 9 for the fourth, 4 for the fifth and 1 for the sixth. However, any choice of six squares can be arrived at in 6! different ways, so the answer is $\frac{36 \cdot 25 \cdot 16 \cdot 9 \cdot 4 \cdot 1}{6!} = 720$.



F07A2 $\mathbf{15+\pi}$. Drawing the 10 lines shown breaks the region R into five 3×1 rectangles and five circular sectors. The central angle of each sector is 72 degrees, so the five sectors together have a central angle of 360 degrees, i.e. they form a complete circle of radius 1. Thus, the total area is $5 \cdot 3 + \pi = 15 + \pi$. The interested reader should consider trying to generalize this result to an arbitrary convex polygon.

F07A3 **15120**. Recall that $\log b^c = c \log b$ and $\log_a b = \frac{\log_x b}{\log_x a} = \frac{\log b}{\log a}$. Thus we can re-write our product as $\left(2\frac{\log 3}{\log 2}\right)\left(3\frac{\log 4}{\log 3}\right)\left(4\frac{\log 5}{\log 4}\right)\left(5\frac{\log 6}{\log 5}\right)\left(6\frac{\log 7}{\log 6}\right)\left(7\frac{\log 8}{\log 7}\right) = 5040\frac{\log 8}{\log 2} = 5040 \cdot 3 = 15120$.

F07A4 $\mathbf{\frac{540}{11}}$. Let t_{40} be the time spent traveling at 40 mph, and likewise for t_{50} and t_{60}. Then we have that $t_{40} = t_{50} + t_{60}$ and that $60t_{60} = 50t_{50} + 40t_{40}$. Substituting from the first equation into the second gives $20t_{60} = 90t_{50}$ so $t_{60} = 9t_{50}/2$. Substituting this back into the first equation gives $t_{40} = 11t_{50}/2$. The value we wish to calculate is $\frac{60t_{60} + 50t_{50} + 40t_{40}}{t_{60} + t_{50} + t_{40}} = \frac{270t_{50} + 50t_{50} + 220t_{50}}{(9t_{50}/2) + t_{50} + (11t_{50}/2)} = \frac{540}{11}$.

F07A5 **-1**. Let the first term of our geometric progression be a and the common ratio be r. Then the third term is ar^2 and the fifth term is ar^4 so we have $ar^4 - ar^2 = ar^2 - a$. Since the progression is non-constant, $a \neq 0$ and we can divide to get $r^4 - r^2 = r^2 - 1$ or in other words $r^4 - 2r^2 + 1 = 0$. The left-hand side of this equation factors as $(r-1)^2(r+1)^2$, so the only possible values for r are ± 1. Since the progression is non-constant, $r \neq 1$ and the answer is -1.

F07A6 $\mathbf{\left(3, 4+2\sqrt{2}\right)}$. Let $\overline{AC}$ intersect the line $x = 1$ at the point $(1, a)$. Then the portion of ΔABC to the left of this line is a right triangle and has area $1/2 \cdot 2 \cdot a = a$. Dropping an altitude from C to the line $x = 1$ shows that the right-hand side has area $1/2 \cdot (s-1) \cdot a$, so we have that $s - 1 = 2$ and $s = 3$. Now, let the line $y = 2$ intersect $\overline{AC}$ at M and $\overline{BC}$ at N. Then $\Delta ABC \sim \Delta MNC$. By the given, the area of ΔMNC is exactly half the area of ΔABC, and because the triangles are similar, the ratio of their areas is the square of the ratio of their altitudes. Thus, $\left(\frac{t-2}{t}\right)^2 = \frac{1}{2}$. Multiplying out, we have $2t^2 - 8t + 8 = t^2$ and so $t = \frac{8 \pm \sqrt{64-32}}{2} = 4 \pm 2\sqrt{2}$. Since $t > 2$ we must take the larger root and $(s,t) = (3, 4+2\sqrt{2})$.

New York City Interscholastic Mathematics League

Senior A Division Contest Number 2

Fall 2007 Solutions

F07A7 **-1/2.** We have that $\cos^2 t+\sin^2 t=1$, so $\sin^2 t=\frac{1}{5}$ and since t is in the first quadrant, $\sin t=5^{-1/2}$. By log and trig rules, $\log_5\cos t+\log_5\tan t=\log_5\cos t\cdot\tan t=\log_5\sin t=\log_5 5^{-1/2}=-1/2$.
Or: $\tan t=\frac{1}{2}$, so we have $\log_5\frac{2}{\sqrt{5}}+\log_5\frac{1}{2}=\log_5 5^{-\frac{1}{2}}=-\frac{1}{2}$.

F07A8 The area of ΔABC is 6. By $K=rs$, the radius of the circle is 1. If we let $CD=x$, then $BD=\sqrt{4^2+(3+x)^2}=\sqrt{25+6x+x^2}$.

By $K=rs$, we obtain: area $\Delta BCD=\frac{1}{2}\cdot x\cdot 4=1\cdot\frac{1}{2}\cdot\left(5+x+\sqrt{25+6x+x^2}\right)$.

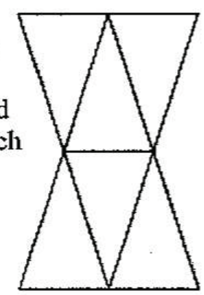
$$4x=5+x+\sqrt{25+6x+x^2}$$
$$3x-5=\sqrt{25+6x+x^2}$$

Squaring both sides yields: $9x^2-30x+25=25+6x+x^2$

$$8x^2-36x=0$$

As $x=0$ is impossible, $x=\frac{9}{2}$.

F07A09 16π. Consider the section through the cones shown at right. An additional segment, corresponding to the circle of intersection of the surfaces of the two cones, is drawn. This circle lies in a plane parallel to and equidistant from the planes that contain the bases of the two cones. Thus, by similar triangles we see that the desired volume consists of two cones of base radius 2 and height 6, joined at their base. One such cone has volume $\frac{\pi}{3}\cdot 2^2\cdot 6=8\pi$, so the volume of the two cones together is 16π.



F07A10 **(3, 1024).** The first few values are $f(0)=3^1$, $f(1)=3f(0)=3^2$ and $f(2)=3f(0)f(1)=3^4$. An educated guess suggests that $f(n)=3^{2^n}$. Then $f(n+1)=3f(0)f(1)\cdots f(n)=3\cdot 3^1\cdot 3^2\cdots 3^{2^n}=3^{1+(1+2+\ldots+2^n)}=3^{2^{n+1}}$, so our guess is confirmed by induction and $f(10)=3^{2^{10}}=3^{1024}$, so $(p,a)=(3,1024)$.

F07A11 We can rewrite the expression as $x=\sqrt{2+\sqrt{6+x}}$. Squaring both sides and subtracting 2 yields $x^2-2=\sqrt{6+x}$. Squaring again yields $x^4-4x^2+4=x+6$. Finally, we get $x^4-4x^2-x-2=0$. Therefore, (h, a, m) = **(-4, -1, -2).**

F07A12 **0, 1, 53.** There are $g\cdot(g-1)\cdot(g-2)$ ways Pascal can choose three green marbles, and $3\cdot 17\cdot g\cdot(g-1)$ ways Pascal can choose one blue and two green marbles. The probabilities of these events are the same if and only if the number of ways they can occur are equal, so we must solve $g(g-1)(g-2)=51g(g-1)$ for integers g. Bringing everything to the left and factoring gives $g(g-1)(g-53)=0$, so g = 0, 1 or 53. (Note that if g = 0 or 1, the probability of both events is equal to 0.)

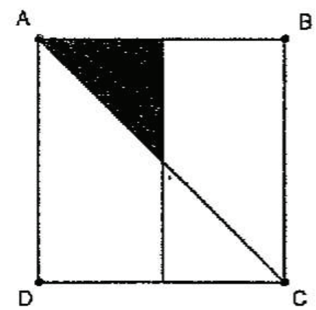
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 3

Fall 2007 Solutions

F07A13 7. The number of diagonals of a regular n-gon is $\frac{n(n-3)}{2}$. Therefore we must solve the equation $2n=\frac{n(n-3)}{2}$. This quadratic equation has solutions $n=0$ and $n=7$. Since there is no such thing as a 0-gon, the answer is 7.

F07A14 22. Notice that $(w+x)+(y+z)=(w+y)+(x+z)=(w+z)+(x+y)$, so our six sums can be broken into three pairs with equal sum. With only five of the six sums, we must still have two pairs with equal sum. The only such pairs among the five sums given are $13+48=25+36=61$, so our desired sum is $61-39=22$.



F07A15 1/8. Choose a point X in R. Since X must be closer to A than to B, X must lie on the same side as A of the perpendicular bisector of segment AB. Since X must be closer to B than to D, X must lie on the same side as B of the perpendicular bisector of segment BD. The region satisfying these requirements is shown: it is an isosceles right triangle with legs of length 1/2 and so with area 1/8. Every point on the interior of this triangle is closer to D than to C, so this region is exactly the region R.

Alternatively, note that every point on the interior of the square fits one of the following criteria: either it is equidistant from two vertices of the square or its distances from the four vertices are distinct. The set of points in the first case have zero total area. For the points in the second case, note that choosing a nearest vertex automatically fixes a furthest vertex: those points in the square nearer to A than any other vertex are all further from C than any other vertex; those nearest B are furthest from D, and so on. Thus, there are four possible choices for the nearest vertex and two possible choices for the next nearest vertex, so a total of 8 distinct regions, one of which is R, whose total area is 1. Because of the symmetry of the situation, each of these regions must have equal area, so in particular the area of R is 1/8.

F07A16 23, 46. Let our integer be $10a+b=ab_{10}=ba_7=7b+a$. Then $3a=2b$ and b is divisible by 3. Note that a and b are digits that can be used in base 7, and both can be used as the left-most digit of an integer, so they are among $\{1, 2, 3, 4, 5, 6\}$. Thus b can be either 3 or 6, and the corresponding values of a are 2 and 4.

F07A17 5. Let $s=\sin^2 15$ so $1-s=\cos^2 15$. Then our expression becomes $6s(1-s)+5(1-s)^2+4s+s^2=(6s-6s^2)+(5-10s+5s^2)+4s+s^2=5$. Note that in particular that the answer is independent of s.

F07A18 2/3. Let p_1 be the probability the machine produces a 1, p_2 the probability it produces a 2, and so on, so $p=p_{10}$ is the quantity we want to maximize. Then $E=7=1\cdot p_1+2\cdot p_2+\ldots+10\cdot p_{10}$. Since all probabilities are positive, we have $7\ge(p_1+p_2+\ldots p_9)+10p_{10}=(1-p_{10})+10p_{10}=9p_{10}+1$. It follows immediately that $p_{10}\le\frac{2}{3}$. We see that 2/3 is actually achieved for a machine that outputs 1 with probability 1/3 and 10 with probability 2/3, so it is the maximum.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 4

Fall 2007 Solutions

F07A19 **5/9.** There are ${}_9C_4 = 126$ total ways to choose four of the nine letters. Of these, ${}_2C_1 \cdot {}_7C_3 = 2 \cdot 35 = 70$ contain exactly one L. Thus, our answer is $70/126 = 5/9$.

F07A20 **5.** In order to calculate the number of 0s at the end of 16! in base 16, we must calculate the number of times that 16 divides $16! = 2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1$. Each factor of 16 adds four factors of 2, so 16^3 is the largest power of 16 which divides 16!, and 16! ends in three 0s base 16. Thus we need a prime which divides 16! exactly three times; from the prime factorization, the only such prime is 5.

F07A21 **6/5.** Let d be the common difference, so $a_2 = a_1 + d$ and $a_4 = a_1 + 3d$. Then we have $a_1 \cdot a_4 = (a_2)^2$, so $a_1 \cdot (a_1 + 3d) = (a_1 + d)^2$ and by expanding and equating like terms, $a_1 d = d^2$. Since the sequence is non-constant, $d \neq 0$ and so $d = a_1 \neq 0$. Thus $a_5 = 5d$ and $a_6 = 6d$ and thus $\frac{a_6}{a_5} = \frac{6}{5}$.

F07A22 **8.** Pairing the first term with the sixth, second with the fifth and third with the fourth term in the product gives $(x^3+1)(x^6+1)(x^3-1)$. Pairing the first and last term in this new product and making the given substitution for x gives the answer of $(3+1)(3-1) = 8$.

F07A23 **4.** Consider two of the chosen subsets, T and U. If T and U have no elements in common, then every element of S belongs to either T or U and so any third three-element subset must intersect at least one of T, U in two elements. Otherwise, every pair of subsets intersect in exactly one point. Now, suppose we have three or more chosen subsets. It is not possible for three of these subsets to share a common element, for if they did then each would need two more elements not in either of the others and so there would be in total at least $1 + 3 \cdot 2 = 7$ elements of S, a contradiction. Thus every element of S belongs to at most two of the chosen subsets. Since each pair of chosen subsets have one element in common and each element belongs to at most one pair of chosen subsets, the number of pairs of subsets must be at most the number of elements. Since we have 6 elements, Gauss cannot choose more than 4 subsets. The choices $S = \{1,2,3,4,5,6\}, T = \{1,2,3\}, U = \{1,4,5\}, V = \{2,4,6\}$ and $W = \{3,5,6\}$ show that four is possible, so it is the maximum.

F07A24 $6\pi + \frac{9\sqrt{3}}{2}$. From the fact that D trisects the altitudes of ΔABC, we have that $DA = DB = DC = \sqrt{3}$. Let O be the center of the circle which passes through B, C, D. Notice that $m\measuredangle BDC = 120^\circ$, so the measure of the (larger) $\widehat{BC}$ is 240° and $m\measuredangle BOC = m\widehat{BDC} = 120^\circ$. Notice that $\Delta BDC \sim \Delta BOC$ by SAS similarity, but since they share side BC we in fact have $\Delta BDC \cong \Delta BOC$ and so quadrilateral $BOCD$ is a rhombus of side length $\sqrt{3}$. The area in question is composed of three of these rhombuses and three 240° sectors of circles. Each rhombus is composed of two equilateral triangles of side length $\sqrt{3}$, and this is also the radius of the circles. Thus, the area in question is

$$3\left(\frac{2}{3} \cdot \pi\left(\sqrt{3}\right)^2\right) + 6\left(\frac{\left(\sqrt{3}\right)^2 \sqrt{3}}{4}\right) = 6\pi + \frac{9\sqrt{3}}{2}.$$

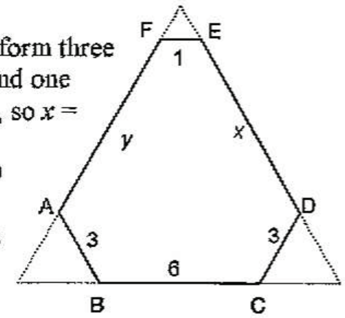
NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division CONTEST NUMBER 5

Fall 2007 Solutions

F07A25 **24.** Note that iTems in packages of three cost less than iTems in packages of five. Thus, Cauchy should seek to purchase as many packages of three as possible. This occurs when Cauchy purchases ten packages of three and one package of five, for a total cost of $10\cdot 2+1\cdot 4=24$ dollars.

F07A26 **8.** Extend sides *BC, DE* and *AF* of the hexagon, as shown. They form three smaller equilateral triangles, one with side length 1 and two with side length 3, and one larger equilateral triangle. The larger triangle has side length $3+6+3=3+x+1$, so $x=8$.



Alternatively, one can extend *EF* and *BC* and draw lines through *A* and *D* perpendicular to these two sides. This inscribes the hexagon in a rectangle, and one can solve the linear equations that arise from ensuring that the opposite sides of the rectangle have the same length to find the length of *DE*.

F07A27 $-\frac{3}{5},\frac{8}{17}$. Solving the given equation for *y* in terms of *x*, we have either that $y=2x$ or that $y=3x/5$. In the first case, we get $\frac{x^2-y^2}{x^2+y^2}=\frac{x^2-4x^2}{x^2+4x^2}=-\frac{3}{5}$; in the second case, we get $\frac{x^2-y^2}{x^2+y^2}=\frac{x^2-9x^2/25}{x^2+9x^2/25}=\frac{8}{17}$.

F07A28 **149.** Note that $50=25\cdot 2$ leaves a remainder of 1 on division by 49. Thus, any integer *n* such that $25n$ leaves a remainder of 1 on division by 49 must differ from 2 by a multiple of 49, i.e. $n=49m+2$ for some positive *m*. Then we seek the smallest *m* such that $49(49m+2)=2401m+98$ leaves a remainder of 1 on division by 25. Then $m+23$ also leaves a remainder of 1 on division by 25; the smallest positive value of *m* is thus 3, and it gives us our answer, $n=49\cdot 3+2=149$.

F07A29 **192.** It would be easy enough to compute the terms of the sequence , however, it is more efficient to notice that $a_{n+2}=a_{n+1}+b_{n+1}=(a_n+b_n)+(a_n-b_n)=2a_n$, so $a_{13}=2a_{11}=2^2a_9=\ldots=2^6a_1=64\cdot 3=192$.

F07A30 **(-3, 1).** A rotation does not alter the distance between the center of rotation and any point, so we must have $\sqrt{a^2+(b-5)^2}=\sqrt{(a+8/5)^2+(b-29/5)^2}$ and $\sqrt{(a-2)^2+(b+9)^2}=\sqrt{(a-26/5)^2+(b+33/5)^2}$. Squaring both of these equalities and collecting like terms gives the two linear equations $2a-b=-7$ and $4a+3b=-9$ which have the unique solution $(a,b)=(-3,1)$.

The angle of rotation is $\arctan\frac{44}{117}$, one of the acute angles in a right triangle with sides of length 44, 117 and 125. It is also the difference between the larger acute angles of a 7-24-25 and a 3-4-5 right triangle.