



New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION
PART I: 10 minutes

CONTEST NUMBER ONE
NYCIML Contest One

SPRING 2007
Spring 2007

- S07B01** The mean of five numbers is 76, the median is 75, the mode is 81, and the range is 11. Compute the value of the second smallest number.
- S07B02** Compute the number of arrangements of the letters in the word *COCOROS* that have three consecutive O's.
-

PART II: 10 minutes

NYCIML Contest One

Spring 2007

- S07B03** Compute the number of positive integers less than 2007 that have an odd number of positive integer factors.
- S07B04** The roots of $x^2 + ax + b = 0$ are additive inverses of each other and also multiplicative inverses of each other. Compute both roots.
-

PART III: 10 minutes

NYCIML Contest One

Spring 2007

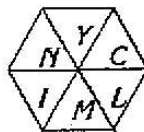
- S07B05** Compute the number of points of intersection of the graphs of the equations $x^2 - y^2 = 0$ and $2x^2 + 3y^2 = 1$.
- S07B06** In $\triangle ABC$, medians $\overline{AD}$ and $\overline{BE}$ intersect at a point F . If $\triangle ABF$ is equilateral and $AB = 10$, compute the perimeter of $\triangle ABC$.
-

ANSWERS

1. 73
2. 60

3. 44
4. $\pm i$

5. 4
6. $10 + 20\sqrt{7}$



New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION
PART I: 10 minutes

CONTEST NUMBER TWO
NYCIML Contest Two

SPRING 2007
Spring 2007

- S07B07 The point $(k, 0)$ is equidistant from the origin and the point $(2, 6)$.
Compute the value of k .
- S07B08 If $\sqrt{x} \sqrt[3]{x} \sqrt[4]{x} = x^h$, compute the value of h .
-

PART II: 10 minutes

NYCIML Contest Two

Spring 2007

- S07B09 A circle is inscribed in rhombus with diagonals of length 6 and 8.
Compute the radius of the circle.
- S07B10 Compute x , if $\log_4(x + 2) = \log_2(2x - 1)$.
-

PART III: 10 minutes

NYCIML Contest Two

Spring 2007

- S07B11 In $\triangle ABC$, point D lies on side $\overline{AC}$ such that $AD : DC = 3 : 2$. Point E lies on side BC such that $CE : EB = 1 : 5$. If the area of $\triangle ABC$ is 75, compute the area of $\triangle CDE$.
- S07B12 The sum of the terms of an infinite geometric progression is $\frac{1}{2}$ and the sum of its first three terms is $\frac{1}{6}$. Compute the common ratio of the geometric progression.
-

ANSWERS

- | | |
|--------------------|-------------------------------|
| 7. 10 | 10. $\frac{5 + \sqrt{41}}{8}$ |
| 8. $\frac{17}{24}$ | 11. 5 |
| 9. $\frac{12}{5}$ | 12. $\frac{\sqrt[3]{18}}{3}$ |



New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION
PART I: 10 minutes

CONTEST NUMBER THREE
NYCIML Contest Three

SPRING 2007
Spring 2007

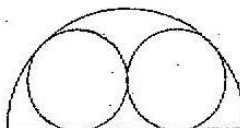
- S07B13** A cube with edge of length 10 is painted purple and is then cut into 1000 unit cubes. Compute the number of these unit cubes that have at least one face painted.
- S07B14** The sum of the first 5 terms of an arithmetic progression is equal to the sum of the 6th and 7th terms. If the fourth term is 70, compute the value of the second term.

PART II: 10 minutes

NYCIML Contest Three

Spring 2007

- S07B15** If $f(x) = ax + b$, and $f(f(f(x))) = 27x + 39$, compute (a, b) .
- S07B16** Two externally tangent circles of radius 5 are inscribed in a semicircle of radius r . Compute r .



PART III: 10 minutes

NYCIML Contest Three

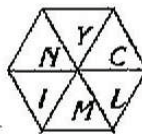
Spring 2007

- S07B17** Compute the value of $(\tan 10^\circ)(\tan 20^\circ)(\tan 30^\circ)(\tan 40^\circ)(\tan 50^\circ)(\tan 60^\circ)(\tan 70^\circ)(\tan 80^\circ)$
- S07B18** Define a *snaking number* to be a positive integer whose decimal representation is of the form $(ABCD)_{10}$, where $A \neq 0$ and $A < C < B < D$. Compute the total number of snaking numbers.

ANSWERS

13. 488
14. 28
15. (3, 3)

16. $5\sqrt{2} + 5$
17. 1
18. 126



New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION
PART I: 10 minutes

CONTEST NUMBER FOUR
NYCIML Contest Four

SPRING 2007
Spring 2007

- S07B19** The arithmetic mean of two positive numbers is 7. Their geometric mean is 4. Compute the sum of their squares.
- S07B20** Compute all values of k such that the equation $x^2 + kx + 2k = 0$ has exactly one real solution.
-

PART II: 10 minutes

NYCIML Contest Four

Spring 2007

- S07B21** If $\sqrt{8+\sqrt{15}} - \sqrt{8-\sqrt{15}} = \sqrt{j}$ for integer j , compute the value of j .
- S07B22** Compute all values of x such that $|2x| + |x+2| = 3$.
-

PART III: 10 minutes

NYCIML Contest Four

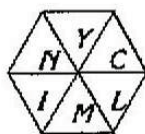
Spring 2007

- S07B23** Compute the sum of the digits of 11^6 .
- S07B24** Let R be the length of the radius of the circumscribed circle, and let r be the length of the radius of the inscribed circle of a triangle with sides of length 8, 8 and 2. Compute R/r .
-

ANSWERS

19. 164
20. 0, 8
21. 2

22. $-1, \frac{1}{3}$
23. 28
24. $\frac{32}{7}$



New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION
PART I: 10 minutes

CONTEST NUMBER FIVE
NYCIML Contest Five

SPRING 2007
Spring 2007

- S07B25** One half of two more than a positive number is twice the reciprocal of the positive number. Compute the number.
- S07B26** Nine distinct lines lie in a plane. Compute the maximum number of points of intersection.
-

PART II: 10 minutes

NYCIML Contest Five

Spring 2007

- S07B27** Compute the value of $\sin\left(2\text{Arc}\cos\frac{5}{13}\right)$.
- S07B28** Each of the numbers 228, 344, and 518 leave the same remainder when divided by the positive integer x . Compute the maximum possible value of x .
-

PART III: 10 minutes

NYCIML Contest Five

Spring 2007

- S07B29** Compute all values of x such that $\sqrt{x-6} = x\sqrt{x-6}$.
- S07B30** Compute the maximum real value of y for which $2x^2 - \sqrt{3}xy + y^2 = 1$ has at least one real solution for x .
-

ANSWERS

- | | | | |
|-----|-------------------|-----|------------------------|
| 25. | $-1 + \sqrt{5}$ | 28. | 58 |
| 26. | 36 | 29. | 1, 6 |
| 27. | $\frac{120}{169}$ | 30. | $\frac{2\sqrt{10}}{5}$ |

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division **CONTEST NUMBER 1**
SPRING 2007 Solutions

1. If we call the numbers a, b, c, d and e such that $a \leq b \leq c \leq d \leq e$, we know that $c = 75$. Moreover, $d = e = 81$. This leads to $a = 70$. The sum of the numbers is $5 \times 76 = 380$. Thus $b = 73$.

2. Imagine that the three O 's were combined to form a new letter O_1 . The number of arrangements of the letters C, C, R, S and O_1 is now $\frac{5!}{2!} = 60$.

3. Any integer having an odd number of positive divisors is a perfect square. Thus, we are looking for the number of perfect squares less than 2007, which is given by $\lfloor \sqrt{2007} \rfloor = 44$.

4. The sum of the roots is $-a$, but is also 0, since they are additive inverses. Thus $a = 0$. The product of the roots is b , but is also 1, since they are multiplicative inverses. Thus $b = 1$. Our quadratic is now just $x^2 + 1 = 0$, and the roots are $\pm i$.

5. Knowing that $x^2 = y^2$ allows us to substitute and obtain $5x^2 = 1 \rightarrow x = \pm \frac{\sqrt{5}}{5}$.

Each value of x in turn gives us two values of y . Therefore, there are **four** points of intersection, namely $\left(\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right); \left(\frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right); \left(-\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$ and $\left(-\frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right)$. Incidentally, the first equation is two intersecting lines (a degenerate hyperbola), and the second equation is an ellipse.

6. $AF = BF = AB = 10$. Since any two medians trisect each other, we also have $AD = \frac{2}{3} AF = 15 = \frac{2}{3} BF = BE$. Let $x = AE$. Using Stewart's Theorem on $\triangle ABE$, we get $(5)(10)(15) + (10)(15)(10) = (10)(5)(10) + 10x^2$. Thus $x^2 = 175$, and we find that $x = 5\sqrt{7}$. By symmetry, $AE = EC = CD = DB = x$. Hence the perimeter of $\triangle ABC$ is $10 + 4x = 10 + 20\sqrt{7}$.

In the alternative, we can establish that $\angle DFB = 120^\circ$, and we can use the law of cosines to obtain $BD = \sqrt{(10)^2 + (5)^2 - 2(10)(5)\cos(120^\circ)} = \sqrt{175} = 5\sqrt{7}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division **CONTEST NUMBER 2**
SPRING 2007 Solutions

7. Using the distance formula, we know that $k = \sqrt{(k-2)^2 + (6-0)^2}$ or $k^2 = k^2 - 4k + 4 + 36$. Solving for k yields $k = 10$.

8. We can begin by rewriting $\sqrt{x \cdot \sqrt[3]{x \cdot \sqrt{x}}} = x^h$ as $\{x[x(x^{\frac{1}{2}})]^{\frac{1}{3}}\}^{\frac{1}{2}} = x^h$ and so $\{x[x^{\frac{3}{2}}]^{\frac{1}{3}}\}^{\frac{1}{2}} = \{x[x^{\frac{1}{2}}]\}^{\frac{1}{2}} = x^{\frac{1}{2}}$. Therefore $h = \frac{1}{2}$.

9. Since the diagonals of a rhombus are perpendicular bisectors of one another, we can compute the side of the rhombus to be 5. The area of the rhombus is now given by $K = \frac{1}{2}d_1d_2$ where d_1 and d_2 denote the diagonals and thus $K = \frac{1}{2}(6 \cdot 8) = 24$. But the area of any quadrilateral circumscribed around a circle is given by $K = rs$ where r denotes the radius of the inscribed circle and s the semiperimeter. Therefore $24 = r \cdot (\frac{4+5}{2})$ and $r = \frac{12}{5}$.

10. Let the common value of the two logarithms be called M . Then the equation may be rewritten as $4^M = x + 2$, $2^M = 2x - 1$. Since 4^M is the square of 2^M , we see that $x + 2 = (2x - 1)^2 = 4x^2 - 4x + 1$. The roots of this quadratic equation are $\frac{1}{5}(5 \pm \sqrt{41})$. Choosing the negative sign yields an x for which $2x - 1 < 0$, producing an undefined logarithm. Thus the unique solution is $x = \frac{1}{5}(5 + \sqrt{41})$.

11. The area of $\triangle CDE$ is given by $\frac{1}{2}(CD)(CE)\sin(\angle ACB)$. The area of $\triangle ABC = 75 = \frac{1}{2}(AC)(BC)\sin(\angle ACB)$. Thus $CD = \frac{2}{5}(AC)$ and $CE = \frac{1}{6}(BC)$. Therefore, the area of $\triangle CDE$ is $\left(\frac{2}{5}\right)\left(\frac{1}{6}\right)(75) = 5$.

12. Let the progression be written $(a, ar, ar^2, ar^3, \dots)$, where the common ratio r satisfies $-1 < r < 1$. The sum of the series is $a/(1-r)$. The sum of the first n terms is $a(1-r^n)/(1-r)$. With the information given, these formulas lead to the equations $2a = 1 - r$ and $6a(1-r^3) = 1 - r$. It follows that $3(1-r^3) = 1$, and therefore $r = \sqrt[3]{\frac{2}{3}} = \frac{\sqrt[3]{18}}{3}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division **CONTEST NUMBER 3**
SPRING 2007 Solutions

13. The unit cubes that have no faces painted are those in the interior of the larger cube. This interior cube has an edge of 8, and thus $8^3 = 512$ cubes that have no faces painted. Therefore, $1000 - 512 = 488$ cubes have at least one face painted.

14. Let $(a, a + d, a + 2d, \dots)$ denote the terms of the arithmetic sequence. Then, according to the given, $a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) = (a + 5d) + (a + 6d)$ which upon simplification leads to $3a = d$. Since the fourth term of the sequence equals 70, we know that $a + 3d = 70$. Substituting $3a$ for d we have $a + 9a = 70$, or $a = 7$. Thus, $d = 3a = 21$ and the second term is $a + d = 28$.

15. Since $f(f(f(x))) = f(f(ax + b)) = f(a(ax + b) + b) = f(a^2x + ab + b)$
 $= a(a^2x + ab + b) + b = a^3x + a^2b + ab + b$ we have that $27x + 39 = a^3x + a^2b + ab + b$
for all real x . So $a^3 = 27$ and $a^2b + ab + b = 39$, and $a = 3$ and $b = 3$.

16. Let the center of the semicircle be O , and let the two circles touch at P . By symmetry, the ray $\overline{OP}$ determines two quarter-circles, each circumscribed about one circle of radius 5. Choose one of these quarter-circles and draw the radius $\overline{OM}$ through the center A of its inscribed circle. Note that $\overline{AM}$ is a radius of the inscribed circle. Therefore $r = OM = OA + AM = 5\sqrt{2} + 5$.

17. For any acute angle θ , $\tan(90^\circ - \theta) = \cot(\theta) = \frac{1}{\tan(\theta)}$. Thus,
 $\tan(90^\circ - \theta) \cdot \tan(\theta) = 1$. Therefore, the product equals 1.

18. Since the four digits are distinct with A being smallest, all four digits must be positive. Now any selection of four digits from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ will produce a unique snaking number. For instance, $\{2, 4, 6, 7\}$ yields 2647. It follows that the number of snaking numbers is ${}_9C_4 = 126$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division CONTEST NUMBER 4
SPRING 2007 Solutions

19. If x and y are our numbers, then we know that $x + y = 14$ and $xy = 16$. Squaring the first expression yields $(x + y)^2 = x^2 + 2xy + y^2 = 196$. Thus,
 $x^2 + y^2 = 196 - 2xy = 196 - 32 = 164$.

20. If $x^2 + kx + 2k = 0$ has exactly one real root r , then r must be a double root. In using quadratic formula to solve for x , the discriminant must equal zero. This yields
 $(k)^2 - 4(1)(2k) = k^2 - 8k = 0$. Thus, $k = 0$ or $k = 8$.

21. Squaring both sides of the equation we have
 $(8 + \sqrt{15}) + (8 - \sqrt{15}) - 2\sqrt{64 - 15} = j$, $j = 2$.

22. Note that if $2x \geq 0$ then also $x + 2 > 0$. In this case both absolute value signs may be deleted, and we have $2x + x + 2 = 3$, or $x = \frac{1}{3}$. The other alternative is $2x \leq 0$. In this case we must solve the two equations $-2x \pm (x + 2) = 3$. This leads to $x = -1$ and $x = -\frac{5}{3}$. However, only $x = -1$ checks in the original equation. Solutions: $\{-1, \frac{1}{3}\}$.

23. One way, of course, is to multiply this out. However, another way is to think of this as the binomial expansion of $(10 + 1)^6$. We can look at the 6th row of Pascal's triangle to compute this: 1 6 15 20 15 6 1 yields the number 1,771,561. The sum of the digits is thus 28. Or we could have just added the digits of the 6th row!

24. Let the vertex angle appear at A , and the base angles at B and C . Let F be the midpoint of BC . By symmetry, the circumcenter O falls on the line AF . Note that $AF = \sqrt{8^2 - 1^2} = \sqrt{63} = 3\sqrt{7}$. Now $\triangle BOF$ is a right triangle with hypotenuse $OB = R$ and legs $BF = 1$ and $OF = AF - AO = 3\sqrt{7} - R$. Using the Pythagorean Theorem once more, $R^2 = 1 + (3\sqrt{7} - R)^2$, so that $R = 32/3\sqrt{7}$. We can compute r from the formula $K = rs$, in which K is the area of the triangle, and s is the semiperimeter. This gives $r = \frac{1}{3}\sqrt{7}$. Finally, $R/r = 32/7$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division **CONTEST NUMBER 5**

SPRING 2007 Solutions

25. The statement gives us the following proportion: $\frac{x+2}{2} = \frac{2}{x}$. This yields the quadratic equation $x^2 + 2x - 4 = 0$. Solving for the positive value of x gives us $x = \frac{-2 + 2\sqrt{5}}{2} = -1 + \sqrt{5}$.

26. Since we are looking for the maximum number of points of intersection, we want the lines to be such that no three are concurrent and no two are parallel. The number of intersection points then becomes ${}_9C_2 = \frac{9!}{2!7!} = \frac{9 \cdot 8}{2 \cdot 1} = 36$.

27. Let $\text{Arccos}(\frac{5}{13}) = \theta$. It then follows that $\cos \theta = \frac{5}{13}$ and $\sin \theta = \frac{12}{13}$. Thus $\sin(2 \cdot \text{Arccos}(\frac{5}{13})) = \sin(2\theta) = 2 \sin \theta \cos \theta = 2 \cdot \frac{12}{13} \cdot \frac{5}{13} = \frac{120}{169}$.

28. The given condition may be written as $228 \equiv 344 \equiv 518 \pmod{x}$. Subtracting 228 from each member yields the equivalent condition $0 \equiv 116 \equiv 290 \pmod{x}$. Thus x is a common divisor of 116 and 290. If x is maximal, then $x = \text{gcd}(116, 290) = 58$.

29. If $x = 6$, then both sides are $= 0$, if $x \neq 6 \rightarrow x = \frac{\sqrt{x-6}}{\sqrt{x-6}} = 1$.

30. Treating the equation as a quadratic in y , and choosing the plus sign to enable a maximum value, we get $y = \frac{1}{2} \left\{ \sqrt{3x} + \sqrt{4-5x^2} \right\}$. We will use an important inequality that deals with sums of square roots: $\sqrt{px} + \sqrt{qy} \leq \sqrt{(p+q)(x+y)}$, with equality if and only if $py = qx$. This can be proven by squaring both sides, subtracting identical terms from both sides, and applying the AM-GM inequality to what remains. In our case, we see that $2y = \sqrt{3x^2} + \sqrt{4-5x^2} \leq \sqrt{(3+5)(x^2 + (\frac{4}{5} - x^2))} = \frac{4}{5} \sqrt{10}$. Thus an upper bound for y is $\frac{2}{5} \sqrt{10}$. Moreover, this bound is attainable, namely when $3(\frac{4}{5} - x^2) = 5x^2$, or $x = \frac{\sqrt{30}}{10}$. Hence the answer is $\frac{2}{5} \sqrt{10}$.

In the alternative, if we let $y = k$ be the maximum value of y , then we obtain $2x^2 - \sqrt{3}xk + k^2 - 1 = 0$. Notice that this is a quadratic in x . Solving for x yields

$$x = \frac{\sqrt{3}k \pm \sqrt{3k^2 - 4(2)(k^2 - 1)}}{4} = \frac{\sqrt{3}k \pm \sqrt{8 - 5k^2}}{4}. \text{ The maximum value of } k \text{ occurs when}$$

$$8 - 5k^2 = 0 \rightarrow k = \frac{2\sqrt{10}}{5}. \text{ Note: this equation is actually the graph of an ellipse.}$$