New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION | **CONTEST NUMBER ONE** | **FALL 2006**

PART I: 10 minutes | **NYCIML Contest One** | **Fall 2006**

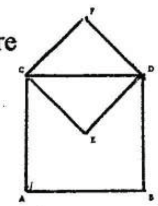
F06B01 In ΔABC, $AC = 7$ and $BC = 9$. If h_A is the height drawn from vertex A and h_B is the height drawn from vertex B, compute $\frac{h_A}{h_B}$.

F06B02 If $k = \frac{(2004)^4 + 4(2004)^3(2) + 6(2004)^2(2)^2 + 4(2004)(2)^3 + (2)^4}{(2008)^3 - 3(2008)^2(2) + 3(2008)(2)^2 - (2)^3}$

Compute k.

PART II: 10 minutes | **NYCIML Contest One** | **Fall 2006**

F06B03 Square $ABDC$ has a side length of 2. $\overline{CD}$ is the diagonal of square $CEDF$ as shown, compute the length of $\overline{AF}$.



F06B04 If $x \cdot x^{\frac{1}{x}} = x^{2x}$ has three real roots, compute x.

PART III: 10 minutes | **NYCIML Contest One** | **Fall 2006**

F06B05 If the lines graphed by the equations $3x - 5y + 4 = 0$ and $2x + ay - 11 = 0$ meet at right angles, compute a.

F06B06 In Pasculand, there are only \$5 and \$11 bills. Compute the largest sum of money that cannot be made using these bills. (Assume that there exists an endless supply of each type of bill.)

<u>ANSWERS</u>

1. $\frac{7}{9}$
2. **2006**
3. $\sqrt{10}$
4. $\pm 1, -\frac{1}{2}$
5. $\frac{6}{5}$
6. **39 (or \$39)**

New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION **CONTEST NUMBER TWO** **FALL 2006**
PART I: 10 minutes **NYCIML Contest Two** **Fall 2006**

F06B07 If $\log(8) \div \log\left(\frac{1}{8}\right) = x$, compute x.

F06B08 If $x+y+z=7$ and $x^2+y^2+z^2=10$, compute $xy+xz+yz$.

PART II: 10 minutes **NYCIML Contest Two** **Fall 2006**

F06B09 A circle is inscribed in a regular hexagon. The perimeter of the hexagon is 36, compute the area of the circle.

F06B10 If $2006jc$ is a 6-digit base 10 number (the digits are $2,0,0,6,j,c$) that is divisible by 72, compute (j, c).

PART III: 10 minutes **NYCIML Contest Two** **Fall 2006**

F06B11 $\frac{a}{b}$ is in lowest terms. If the denominator of the fraction is added to both the numerator and the denominator, the fraction is doubled. Compute $\frac{a}{b}$.

F06B12 If all the 6-digit numbers formed by using the digits 1, 2, 3, 4, 5, and 6, without repetition, are listed from least to greatest, compute the 500th number.

ANSWERS

7. **-1**
8. $\mathbf{\frac{39}{2}}$
9. $\mathbf{27\pi}$
10. **(6, 4)**
11. $\mathbf{\frac{1}{3}}$
12. **516243**

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER THREE** **FALL 2006**

PART I: 10 minutes **NYCIML Contest Three** **Fall 2006**

F06B13 Compute all real x: $x+\sqrt{x-5}=7$.

F06B14 Compute the unit's digit of: $1^5+2^5+3^5+4^5+5^5+6^5+\ldots+2006^5$.

PART II: 10 minutes **NYCIML Contest Three** **Fall 2006**

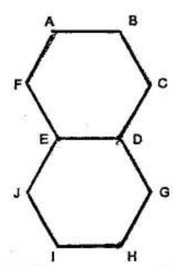
F06B15 In ΔABC, $\sin(A)=3/5$ and $\sin(B)=4/5$. Compute the value of $\sin(C)$.

F06B16 Larry and Gill are going to play a game. They put cards with the letters *C, O, R, N, E, L,* and *L* in a hat (each of the seven cards has one of the letters and there are two cards with an *L*.) They will alternate picking cards, one at a time, without replacement, until someone wins by picking an *L*. (They do not see the card they pick until they pick it.) If Larry picks first, compute the probability that Larry will win.

PART III: 10 minutes **NYCIML Contest Three** **Fall 2006**

F06B17 If $3x^2-3x+h$ is divisible by $x-4$, compute h.

F06B18 *ABCDEF* and *EDGHIJ* are regular hexagons as shown. If $AB=2$, compute AH.



ANSWERS

13. **6**
14. **1**
15. **1**
16. $\mathbf{\frac{4}{7}}$
17. **-36**
18. $\mathbf{2\sqrt{13}}$

SENIOR B DIVISION **CONTEST NUMBER FOUR** **FALL 2006**

PART I: 10 minutes **NYCIML Contest Four** **Fall 2006**

F06B19 If $i=\sqrt{-1}$, and the value of $(1-i)^{10}=a+bi$ where a and b are integers, compute $(a,\ b)$.

F06B20 The average (arithmetic mean) of a set of h numbers is m. The average of a different set of m numbers is h. If the average of all the numbers is p, express $\frac{1}{h}+\frac{1}{m}$ in terms of p, in simplest form.

PART II: 10 minutes **NYCIML Contest Four** **Fall 2006**

F06B21 Compute the number of integers from 1 to 2006 inclusive that are not divisible by 3 or 7.

F06B22 In ΔABC, point D lies on $\overline{AC}$ such that $AD<DC$. If $AB=20, AC=30$, $BD=13$, and the area of $\Delta ABC=180$, compute the area of ΔBDC.

PART III: 10 minutes **NYCIML Contest Four** **Fall 2006**

F06B23 Siggy tosses 2 fair coins. Siggy shows you one of the coins, and it is "heads". Compute the probability that the other coin is "tails"?

F06B24 Compute all ordered triples (x, y, z) of integers:

$$x+yz=6$$
$$y+xz=6$$
$$z+xy=6$$

ANSWERS

19. **(0, -32)**
20. $\mathbf{\frac{2}{p}}$
21. **1147**
22. **114**
23. $\mathbf{\frac{2}{3}}$
24. **(1, 1, 5), (1, 5, 1), (5, 1, 1), (2, 2, 2), and (-3, -3, -3)**

New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION **CONTEST NUMBER FIVE** **FALL 2006**
PART I: 10 minutes **NYCIML Contest Five** **Fall 2006**

F06B25 Jimmy has a 10 ounce drink that is 60% coffee. Jimmy adds milk until his drink is 40% coffee. Compute the amount of milk, in ounces, that he added.

F06B26 If $h = 11!$, compute the number of positive integral divisors of h.

PART II: 10 minutes **NYCIML Contest Five** **Fall 2006**

F06B27 Helen rolls one fair six-sided die. Jim rolls two fair six-sided dice. Compute the probability that the sum of the numbers showing on Jim's dice equals the number showing on Helen's die.

F06B28 If $x^3 + 2x^2 + kx + 3 = 0$ has at least one rational root, compute all possible values of k.

PART III: 10 minutes **NYCIML Contest Five** **Fall 2006**

F06B29 If $4 - \dfrac{3}{2 + \dfrac{x}{1-x}} = \dfrac{a-x}{b-x}$, $x \neq 1, 2$, a and b are integers, compute (a, b).

F06B30 In rectangle $ABCD$, point E is the intersection of the two diagonals, point G lies on $\overline{BC}$, and point F is the intersection of $\overline{AC}$ and $\overline{DG}$. If $AB = 40$, $BC = 30$, and $BG = 10$, compute EF.

<u>**ANSWERS**</u>

25. **5**
26. **540**
27. $\mathbf{\dfrac{5}{72}}$
28. **-16, -6, -2 and 4**
29. **(5, 2)**
30. **5**

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER ONE** **FALL 2006**

Solutions

1. For any triangle, $Area = \frac{1}{2}(base)(height)$. As such, any triangle has three *base-height* pairs. So, $A_{\Delta ABC} = \frac{1}{2}(AC)(h_B) = \frac{1}{2}(BC)(h_A)$. Thus, $\frac{(h_A)}{(h_B)} = \frac{(AC)}{(BC)} = \mathbf{\frac{7}{9}}$.

2. $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ and $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.
Thus, $k = \frac{(2004+2)^4}{(2008-2)^3} = \frac{(2006)^4}{(2006)^3} = \mathbf{2006}$.

3. Construct midpoint G to $\overline{AB}$ and draw segments $\overline{GF}$ and $\overline{AF}$. $EF = 2$ and $GE = \frac{1}{2}(EF) = 1$. Thus, $GF = 3$ and $AG = 1$. By the Pythagorean Theorem, $AF = \sqrt{(AG)^2 + (GF)^2} = \mathbf{\sqrt{10}}$.

4. We can simplify the exponent on the left hand side as follows:
$x \cdot x^{\frac{1}{x}} = x^1 \cdot x^{\frac{1}{x}} = x^{1+\frac{1}{x}} = x^{\frac{x+1}{x}}$. Thus, by equating the exponents, we get $\frac{x+1}{x} = 2x \rightarrow x+1 = 2x^2$. Solving this quadratic yields $x = -\frac{1}{2}$ and $x = 1$. The question said that there are 3 real roots, so a careful look at the problem leads us to see that $x = -1$ also works. Thus the roots are $\mathbf{\pm 1, -\frac{1}{2}}$.

5. If the two lines meet at right angles, their slopes are negative reciprocals. Putting both lines into slope-intercept form yields $y = \frac{3}{5}x + \frac{4}{5}$ and $y = -\frac{2}{a}x + \frac{11}{a}$.
So, $\frac{3}{5} = \frac{a}{2} \rightarrow a = \mathbf{\frac{6}{5}}$.

6. Let x = number of \$5 bills, and y = number of \$11 bills. One way to proceed is to realize that there are two ways to make \$55--$(11,0)$ and $(0,5)$. Moreover, we can "manufacture" \$1 by subtracting two \$5's and adding one \$11. Thus, \$56, \$57, \$58, and \$59 are $(9,1)$, $(7,2)$, $(5,3)$, and $(3,4)$ respectively. We can now make \$60 out of \$5's entirely and then proceed to all higher multiples of 5 in a similar manner. In fact, the same process can begin at \$45 with $(9,0)$ and at \$40 with $(8,0)$. A problem happens at \$35 with $(7,0)$. There are not enough \$5's to exchange to create \$39 which would have the solution $(-1,4)$. Thus, \$39 is the answer.

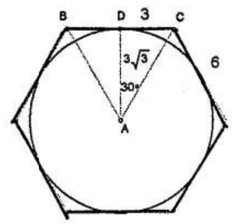
New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION **CONTEST NUMBER TWO** **FALL 2006**

Solutions

7. $\log(8) \div \log\left(\frac{1}{8}\right) = \dfrac{\log(8)}{-1 \cdot \log(8)} = \mathbf{-1}$.

8. $(x+y+z)^2 = 49 = x^2 + y^2 + z^2 + 2(xy + xz + yz)$. Thus, $49 = 10 + 2(xy + xz + yz)$. This yields $xy + xz + yz = \dfrac{\mathbf{39}}{\mathbf{2}}$.



9. Since the perimeter of the hexagon is 36, each side is 6. $\overline{AD}$ is an angle bisector, a median and an altitude to $\overline{BC}$. Thus $CD = 3$ and triangle ABC is a 30, 60, 90 triangle. $AD = 3\sqrt{3}$. The area of the circle is $\pi \cdot \left(3\sqrt{3}\right)^2 = \mathbf{27\pi}$.

10. If a number is divisible by 72, then it must satisfy divisibility by 8 and 9. For divisibility by 8, the last three digits of the number must form a 3-digit number that is a multiple of 8. For divisibility by 9, the sum of the digits of the number must be a multiple of 9 therefore $8 + j + c = 9$ or $18 \rightarrow j + c = 1$ or 10. To test to see if *6jc* is a multiple of 8, the only numbers we must try to divide by 8 are 682, 664, 646, 628, 610. Only 664 satisfies divisibility by 8 and the answer is **200664**.

11. $\dfrac{a+b}{2b} = \dfrac{2a}{b} \rightarrow \dfrac{a+b}{2a} = \dfrac{2b}{b} \rightarrow \dfrac{a+b}{2a} = 2 \rightarrow a+b = 4a \rightarrow b = 3a \rightarrow \dfrac{a}{b} = \dfrac{\mathbf{1}}{\mathbf{3}}$.

12. There are a total of 6! = 720 permutations of these digits. Of these, 5! = 120 begin with "1", 120 begin with "2", and so forth. Thus, the last number that begins with "4" is the 480th number. 4! = 24 numbers begin with "51", so the 504th number is 516432. There are 3! = 6 numbers that begin with "512", 6 numbers that begin with "513", and 6 numbers that begin with "514". Thus the 498th number is 514632. 499th is 516234. 500th is **516243**.

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER THREE** **FALL 2006**

Solutions

13. Rearranging the equation we get $\sqrt{x-5} = 7 - x$. Squaring both sides gives $x^2 - 15x + 54 = 0$. This yields two solutions, $x = 6$ and $x = 9$. Only $x = \mathbf{6}$ satisfies the original equation.

14. It can be shown that the unit's digit of x^5, where x is a base-10 number, is the same as the unit's digit of x. Thus, $1^5 + 2^5 + 3^5 + 4^5 + 5^5 + 6^5 + \ldots + 2006^5$ has the same unit's digit as $1 + 2 + 3 + 4 + \cdots + 9 + 0 + 1 + 2 + \ldots + 5 + 6$. The unit's digit of $1 + 2 + 3 + 4 + \cdots + 9 + 0$ is 5. Thus, the unit's digits of the sum every 20 consecutive integers is 0. As a result, the sum of the first 2000 numbers has a unit's digit of 0. Therefore, the unit's digit of the sum in this problem is just the unit's digit of 1 + 2 + 3 + 4 + 5 + 6, which is **1**.

15. In a 3-4-5 right triangle, the value of the sine of the acute angles are 3/5 and 4/5. Thus, $\angle A$ and $\angle B$ are complementary. $\angle C$ is a right angle, and $\sin(C) = \mathbf{1}$.

16. Larry can win on the 1st pick, the 3rd pick, or the 5th pick. Larry would never have the opportunity to have a 7th pick because if Gill had an opportunity to have a turn on the 6th pick, he would be guaranteed to pick an "*L*" because the two remaining letters would both be "*L*". Thus, *P*(Larry wins) =

$$\left(\frac{2}{7}\right) + \left(\frac{5}{7}\right)\left(\frac{4}{6}\right)\left(\frac{2}{5}\right) + \left(\frac{5}{7}\right)\left(\frac{4}{6}\right)\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{2}{3}\right) = \frac{12}{21} = \mathbf{\frac{4}{7}}.$$

17. One way to attack this problem is to use the Remainder Theorem. Thus, if $f(x) = 3x^2 - 3x + h$ is divisible by $x - 4$, then $f(4) = 3(4)^2 - 3(4) + h = 0$. Solving for h yields $h = \mathbf{-36}$.

18. After we compute AI, we can use the Pythagorean Theorem in ΔAIH to find AH. Since $AI = 2(AE)$, we need to find AE. To accomplish this, we can use the law of cosines and the fact that $m\angle AFE = 120°$.

$$(AF)^2 + (FE)^2 - 2(AF)(FE)\cos(120°) = (AE)^2$$

Substituting for AF and FE yields $(2)^2 + (2)^2 - 2(2)(2)\left(-\frac{1}{2}\right) = (AE)^2$. Therefore,

$AE = 2\sqrt{3}$ and $AI = 4\sqrt{3}$. We now use $(AI)^2 + (IH)^2 = (AH)^2$. Thus,

$$AH = \sqrt{\left(4\sqrt{3}\right)^2 + (2)^2} = \sqrt{52} = \mathbf{2\sqrt{13}}.$$

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER FOUR** **FALL 2006**

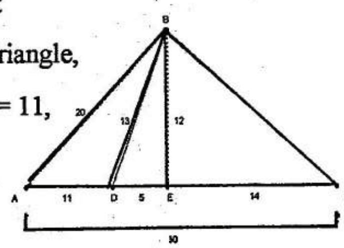
Solutions

19. $(1-i)^2 = 1-2i+i^2 = -2i$. Thus, $(1-i)^{10} = (-2i)^5 = -32i^5 = -32i$. Therefore, $a = 0$, and $b = -32$ and the answer is **(0, -32)**.

20. The sum of each set of numbers is hm; therefore, the sum of all $h + m$ numbers is $2hm$. So, $p = \frac{2hm}{h+m}$. We can now obtain $\frac{1}{p} = \frac{h+m}{2hm} = \frac{1}{2}\left(\frac{1}{m} + \frac{1}{h}\right)$. Thus, $\frac{1}{h} + \frac{1}{m} = \frac{2}{p}$.

21. There are $\left\lfloor \frac{2006}{3} \right\rfloor = 668$ multiples of 3 less than or equal to 2006, and $\left\lfloor \frac{2006}{7} \right\rfloor = 286$ multiples of 7. However, there are $\left\lfloor \frac{2006}{21} \right\rfloor = 95$ multiples of 21 that have been counted in both lists. Therefore, there are $668 + 286 - 95 = 859$ numbers that have 3 or 7 as a factor, and as a result, there are $2006 - 859 =$ **1147** numbers that have neither 3 nor 7 as a factor. ($\lfloor x \rfloor$ represents the greatest integer less than or equal to x.)

22. Draw altitude $\overline{BE}$. From the area of ΔABC, we know that $180 = \frac{1}{2}(AC)(BE) = 15(BE)$. Thus, $BE = 12$. ΔBDE is a right triangle, so $DE = 5$. ΔBAE is a right triangle, so $AE = 16$. Therefore, $AD = 11$, and $DC = 19$. Thus, the area of $\Delta BDC = \frac{1}{2}(19)(12) =$ **114**.



Note, that if point D were between points E and C, then AD would equal 21 and DC would equal 9, thereby violating the constraint requiring $AD < DC$.

23. The possible outcomes are HH, HT, TH, and TT. We know that one of the two coins is H, so the outcome TT is impossible. Of the three outcomes remaining, two have "tails". Thus, the probability is $\mathbf{\frac{2}{3}}$.

24. If we subtract the first two equations, we obtain $x - y + yz - xz = (x-y)(1-z) = 0$. Thus, either $x = y$ or $z = 1$. If $x = y$, then: $\begin{matrix} x + xz = 6 \\ z + x^2 = 6 \end{matrix}$. Substituting for z yields $x + x(6 - x^2) = 6$ or $x^3 - 7x + 6 = 0$. We know x must be an integer, so we can use the rational roots theorem to test the eight possibilities, namely $\pm1, \pm2, \pm3, \pm6$. Of these, $x = 1$, 2 and -3 work. This leads to solutions $(1,1,5)$, $(2,2,2)$ and $(-3,-3,-3)$. Noting that the original system has symmetry, we can conclude that the remaining two solutions are $(1,5,1)$ and $(5,1,1)$. We could also get these remaining two solutions by using $z = 1$ and solving for x and y. The required triples are: **(1, 1, 5), (1, 5, 1), (5, 1, 1), (2, 2, 2), and (-3, -3, -3).**

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER FIVE** **FALL 2006**

Solutions

25. Currently, Jimmy has 6 oz of coffee. If x is the amount of milk added, then: $\frac{6}{10+x}=\frac{2}{5}\rightarrow 30=20+2x\rightarrow x=\mathbf{5}$.

26. To determine the number of factors, we need the prime factorization of 11!. This yields $2^8\cdot 3^4\cdot 5^2\cdot 7^1\cdot 11^1$. Thus, there are $(8+1)(4+1)(2+1)(1+1)(1+1)=\mathbf{540}$.

27. Helen can have outcomes 1, 2, 3, 4, 5 and 6. Jim can have outcomes 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. Thus, the only overlaps are 2, 3, 4, 5 and 6. The probability of a match is therefore P(both 2) + P(both 3) + P(both 4) + P(both 5) + P(both 6) =
$\left(\frac{1}{6}\right)\left(\frac{1}{36}\right)+\left(\frac{1}{6}\right)\left(\frac{2}{36}\right)+\left(\frac{1}{6}\right)\left(\frac{3}{36}\right)+\left(\frac{1}{6}\right)\left(\frac{4}{36}\right)+\left(\frac{1}{6}\right)\left(\frac{5}{36}\right)=\frac{15}{216}=\mathbf{\frac{5}{72}}$.

28. If the polynomial has a rational root, and k is an integer, then the rational roots theorem states that the only possible rational roots are ± 1 and ± 3. Substituting these values for x, the possible values of k are **-6, 4, -16 and -2**.

29. $4-\frac{3}{2+\frac{x}{1-x}}=4-\frac{3-3x}{2-2x+x}=4-\frac{3-3x}{2-x}=\frac{8-4x}{2-x}-\frac{3-3x}{2-x}=\frac{5-x}{2-x}=\frac{a-x}{b-x}$. Thus $(a,b)=(\mathbf{5,2})$.

30. ΔCGF is similar to ΔADF. Therefore, $\frac{GC}{AD}=\frac{CF}{AF}$. $AC=50$ so $AE=EC=25$. Let $EF=x$, then the proportion becomes $\frac{20}{30}=\frac{25-x}{25+x}\rightarrow 50+2x=75-3x\rightarrow x=\mathbf{5}$.