New York City Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Sophomore-Freshman Division

CONTEST NUMBER 1

PART I *SPRING 2006* *CONTEST 1* *TIME: 10 MINUTES*

S06SF1 Compute the smallest positive integral factor of 3476 that contains two digits.

S06SF2 $5\sqrt{x}-x-6=0$. Compute all real x.

PART II *SPRING 2006* *CONTEST 1* *TIME: 10 MINUTES*

S06SF3 Sho the Great was born and died, respectively, in years that were the perfect squares of consecutive positive integers. If he died on his birthday at the age of 55, compute the year he was born.

S06SF4 If r is one more than the sum of the squares of three consecutive odd integers; compute the greatest common divisor of all possible r.

PART III *SPRING 2006* *CONTEST 1* *TIME: 10 MINUTES*

S06SF5 If $\frac{x+2}{x+1}$ and its reciprocal are both integers, compute x.

S06SF6 Gee Hoon has ten cards, labeled consecutively 1, 2, 3,...,10. He chooses three of the cards and gets a sum of 14. Compute the number of different sets of cards for which this could happen.

ANSWERS:

S06SF1	**11**
S06SF2	**4, 9**
S06SF3	**729**
S06SF4	**12**
S06SF5	$\mathbf{-\frac{3}{2}}$
S06SF6	**9**

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CONTEST NUMBER 2

PART I *SPRING 2006* *CONTEST 2* *TIME: 10 MINUTES*

S06SF7 Three positive integers are added two at a time giving sums of 105, 128, and 141. Compute the smallest integer.

S06SF8 In an isosceles right triangle, the difference between the length of the hypotenuse and the length of a leg is 5. Compute the length of the hypotenuse.

PART II *SPRING 2006* *CONTEST 2* *TIME: 10 MINUTES*

S06SF9 Brad and Angela are planning their wedding. There is a charge to rent the room where they will have the party plus an additional charge per guest. If there are 400 people at the party, the total cost will be \$130,000. If there are 280 people at the party, the total cost will be \$97,000. Compute the cost of the party if 100 people attend.

S06SF10 Paul is older than 9 and younger than 100. If he subtracts the product of the digits of his age from his age, he gets 19. Compute all the possible ages that Paul can be.

PART III *SPRING 2006* *CONTEST 2* *TIME: 10 MINUTES*

S06SF11 Three cuts are made through a large cube to create 8 identical smaller cubes. If the total surface area of the large cube is 13, compute the total surface area of the small cubes.

S06SF12 The planet Jostern has a year that contains only 8 days. If three residents of Jostern are in a room, compute the probability that at least two were born on the same day of the year.

ANSWERS:

S06SF7	**46**
S06SF8	$5\sqrt{2}+5$
S06SF9	**\$47,500**
S06SF10	**21, 47**
S06SF11	**26**
S06SF12	$\frac{11}{32}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Sophomore-Freshman Division

CONTEST NUMBER 3

PART I *SPRING 2006* *CONTEST 3* *TIME: 10 MINUTES*

S06SF13 The Chess club shares its winnings. The first place finisher gets half of the money, the second place finisher gets 3/4 of the remainder of the money, then the third place finisher gets 3/4 of the money that now remains, and then the two fourth place finishers equally share what is left. If the fourth place finishers each got \$5, compute the total amount of money that was won by the club members.

S06SF14 $(2x+3)^{x^2-3x+2}=1$. Compute all possible real values of x for which this is true.

PART II *SPRING 2006* *CONTEST 3* *TIME: 10 MINUTES*

S06SF15 If k is a natural number and $k(k+1)$ is divided by 7, compute all possible remainders.

S06SF16 Two sides of a triangle measure 2 and 7 and the length of the third side of the triangle is the same as the area of the triangle (without regards to units). Compute the area of the triangle.

PART III *SPRING 2006* *CONTEST 3* *TIME: 10 MINUTES*

S06SF17 Compute: $\dfrac{2006!-2005!}{2004!}$.

S06SF18 The number of water lilies in a pond increases at a constant rate. Mrs. Carp released fish into the pond and after 96 days, they have eaten all of the water lilies (those that were there at first and those that grew in the 96 days.) The fish eat the water lilies at a constant rate. 70 fish would have taken 24 days to eat all of the lilies and 30 fish would have taken 60 days to eat all of the lilies, compute the number of fish that Mrs. Carp released into the pond.

ANSWERS:

S06SF13	**\$320**
S06SF14	**1,2,-1,-2**
S06SF15	**0,2,5,6**
S06SF16	**$3\sqrt{5}$**
S06SF17	**4020025**
S06SF18	**20**

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NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Sophomore-Freshman Division CONTEST NUMBER 1
Spring 2006 Solutions

S06SF1 $3476 = 4 \cdot 869 = 4 \cdot 11 \cdot 79$. The smallest factor above 10 is **11**.

S06SF2 If we define a new variable y such that $y = \sqrt{x}$, the equation becomes $y^2 - 5y + 6 = 0$. The solutions to this equation are 2 and 3, which correspond to values of **4 and 9** for x.

S06SF3 Let x^2 be the year that Sho the Great was born.
$(x+1)^2 - x^2 = 55 \rightarrow 2x + 1 = 55 \rightarrow x = 27$. x^2=**729**

S06SF4 The smallest possible value of r is when the three numbers are -1, 1, and 3, so that r=12. After this, the three numbers can each be increased by 2 to get the next triple ($n, n+2, n+4$ becomes $n+2, n+4, n+6$), which is the same thing as increasing the smallest number by 6. Increasing a number n by 6 increases its square by $12n+36$, which is a multiple of 12. Thus all possible values of r are divisible by 12, and since the first one is exactly 12, their greatest common divisor must be exactly **12**.

S06SF5 The only integers whose reciprocals are also integers are 1 and -1. If $\frac{x+2}{x+1} = 1 \rightarrow x+2 = x+1$ and there are no solutions. If $\frac{x+2}{x+1} = -1 \rightarrow x+2 = -x-1 \rightarrow x = -\frac{3}{2}$.

S06SF6 An orderly counting gives us:
$(10,1,3), (9,1,4), (9,2,3), (8,1,5), (8,2,4), (7,1,6), (7,2,5), (7,3,4), (6,3,5)$ for a total of **9**.

New York City Interscholastic Mathematics League

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Sophomore-Freshman Division CONTEST NUMBER 2
Spring 2006 Solutions

S06SF7 $a+b=105;\ b+c=128; a+c=141$. Therefore $a+b+c=\frac{105+128+141}{2}=187$. Subtract $a+c$, the largest of the pairwise sums, to obtain $b=\mathbf{46}$.

S06SF8 If the length of the leg is x, the length of the hypotenuse is $x\sqrt{2}$. We now have $x\sqrt{2}-x=5\rightarrow x\left(\sqrt{2}-1\right)=5\rightarrow x=\frac{5}{\sqrt{2}-1}$. rationalizing the denominator gives us: $5\sqrt{\mathbf{2}}+\mathbf{5}$.

S06SF9 When the number of people at the party goes down by 120, the cost decreases by \$33,000. So if the number of people goes down by a further 180, the cost will drop by $\$33,000\cdot\frac{180}{120}=\$49,500$, yielding a final answer of **\$47,500**.

S06SF10 Since Paul's age must be two digits, we can represent it as $10t+u$.
$10t+u-tu=19\rightarrow t(10-u)+u=19$. An even number for u would lead to no solutions, and testing odd digits gives us **21 , 47**.

S06SF11 Each small cube has half the edge length and therefore one quarter the surface area of the large cube. Since there are 8, the total surface area is twice 13, or **26**.

S06SF12 Each resident can have one of eight birthdays, giving $8\cdot8\cdot8=512$ possibilities. There are ${}_8C_3=56$ possible ways of choosing three different birthdays, giving $6\cdot56=336$ possible ways for three people to have these (different) birthdays. All $512-336=176$ other birthday sets will have at least one shared birthday. Thus the probability is: $\frac{176}{512}=\frac{\mathbf{11}}{\mathbf{32}}$.

OR

The probability that resident two does not have the same birthday as resident one is $\frac{7}{8}$. The probability that resident three does not have the same birthday as resident one or resident two is $\frac{6}{8}$. Therefore, the probability that no two have the same birthday is $\frac{6}{8}\cdot\frac{7}{8}=\frac{21}{32}$. Therefore, the probability that at least two have the same birthday is $1-\frac{21}{32}=\frac{\mathbf{11}}{\mathbf{32}}$.

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NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Sophomore-Freshman Division CONTEST NUMBER 3
Spring 2006 Solutions

S06SF13 The second place finisher gets ¾ of a half of the winnings, or 3/8, leaving 1/8. The third place finisher therefore gets ¾ of 1/8 of the winnings, leaving 1/32, so that each fourth place finisher gets $\frac{1}{64}$ of the money. Thus the total amount is $\$5 \cdot \frac{64}{1} = \mathbf{\$320}$.

S06SF14 This can be true if $2x+3=1$, or $2x+3=-1$ and x^2-3x+2 is even, or if $x^2-3x+2=0$ (as long as $2x+3 \neq 0$). For the first case, $x=-1$, for the second case $x=-2$ and the exponent is even, and for the third case $x=1$ or $x=2$. Thus the answers are **1,2,-1,-2**

S06SF15 We look mod 7: $0 \cdot 1 = 0$; $1 \cdot 2 = 2$; $2 \cdot 3 = 6$; $3 \cdot 4 = 5$; $4 \cdot 5 = 6$; $5 \cdot 6 = 2$.. Thus the remainder can be **0, 2, 5, 6.**

S06SF16 Call the length of the third side and the area x. If h is the altitude to the triangle with base x, then the area of the triangle is $\frac{1}{2}hx = x \rightarrow h = 2$. Since the side adjacent to x is also 2, the triangle is a right triangle. Thus, $x^2+2^2=7^2 \rightarrow x = 3\sqrt{5}$.

S06SF17 $\frac{2006!-2005!}{2004!} = \frac{2005!(2006-1)}{2004!} = 2005 \cdot 2005 = \mathbf{4020025}$.

S06SF18 Let l be the number of water lilies that one fish eats in one day. In 24 days, 70 fish would eat $70 \cdot 24l = 1680l$ water lilies, in 60 days, 30 fish would eat $60 \cdot 30l = 1800l$ water lilies. We thus see that $1800l - 1680l = 120l$ water lilies grew within $60 - 24 = 36$ days. So within 24 days, $120l \cdot \frac{24}{36} = 80l$ water lilies grew. We now see that there were $1680l - 80l = 1600l$ water lilies present when Mrs. Carp released the fish. Within 96 days, $96 \cdot \frac{120l}{36} = 320l$ water lilies grew. Therefore the fish must eat $1600l + 320l = 1920l$ water lilies. In one day, the fish eat $\frac{1920l}{96} = 20l$ water lilies. This would take $\frac{20l}{l} = \mathbf{20}$ fish.