New York City Interscholastic Mathematics League

SENIOR B DIVISION | **CONTEST NUMBER ONE** | **SPRING 2006**

PART I: 10 minutes | **NYCIML Contest One** | **Spring 2006**

S06B01. A club will obtain a ratio of men to women of 3 to 1 if it <u>either</u> increases the number of men by 18 <u>or</u> decreases the number of women by y. Compute the value of y.

S06B02. If $xy + 2x + y = 25$, and x and y are both positive integers, compute all ordered pairs (x,y).

PART II: 10 minutes | **NYCIML Contest One** | **Spring 2006**

S06B03. A right circular cone with height 10 is divided into a smaller right circular cone and a frustum by a plane parallel to the base of the larger cone. If the height of the smaller cone is 4, and if the volume of the smaller cone is 12π, compute the radius of the larger cone.

S06B04. What is the remainder when 2^{2005} is divided by 17?

PART III: 10 minutes | **NYCIML Contest One** | **Spring 2006**

S06B05. Compute the area of the smallest circle passing through the points (2, 7) and (8, -1)?

S06B06. A "Helen" sequence is defined as follows: The first two terms, H_1 and H_2 are positive integers a and b respectively such that $a < b$. Each term thereafter is generated by taking the sum of the previous two terms. For instance, if $H_1 = 3$ and $H_2 = 7$, then $H_3 = 3 + 7 = 10$, $H_4 = 7 + 10 = 17$, $H_5 = 10 + 17 = 27$, etc. In this example, $a = 3$ and $b = 7$.

If the <u>sum</u> of the first ten terms of a Helen sequence is 462, compute the ordered pair (a, b).

ANSWERS

1.	6	3.	15/2	5.	25π
2.	(8, 1) and (2, 7)	4.	15	6.	(2, 4)

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SENIOR B DIVISION | **CONTEST NUMBER TWO** | **SPRING 2006**

PART I: 10 minutes | **NYCIML Contest Two** | **Spring 2006**

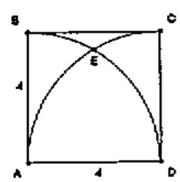
S06B07. If the three points $(3, k)$, $(k, 7)$ and $(2k, k - 5)$ are all collinear, compute all possible real values of k.

S06B08. If 10^k is a factor of $(5^3)!$, compute the largest possible integer value for k.

PART II: 10 minutes | **NYCIML Contest Two** | **Spring 2006**

S06B09. Simplify the following expression into one containing no trigonometric functions and in simplest form. $\dfrac{\sin(15°)+\cos(15°)}{\sin(75°)+\cos(75°)}$

S06B10. In the following diagram, two quarter circles are inscribed in a square $ABCD$ with side length 4. The area of the region BCE bounded by the quarter circles and side BC is given by $p - q\sqrt{3} - r\pi$. If p, q and r are rational numbers, compute (p, q, r).



PART III: 10 minutes | **NYCIML Contest Two** | **Spring 2006**

S06B11. The base-nine number 325_9 is equal to the base-b number 222_b where b is a positive integer. Compute the value of b.

S06B12. A parallelogram has side lengths of 4 and 6. If the length of the longer diagonal is 8, compute the length of the shorter diagonal?

ANSWERS

7.	k = 9, 2	10.	(16, 4, 8/3)
8.	31	11.	11
9.	1	12.	$2\sqrt{10}$

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SENIOR B DIVISION | **CONTEST NUMBER THREE** | **SPRING 2006**

PART I: 10 minutes | **NYCIML Contest Three** | **Spring 2006**

S06B13. Helen works at twice the rate of Jim. Sigmund works at twice the rate of Helen. If working together Helen, Jim and Sigmund can do a job in 1 hour, compute the number of hours it would take Jim to do the job if he worked alone?

S06B14. Given the system of equations:

$$x + 2y = 5$$
$$2x + z = 8$$
$$y + 2z = 2$$

Compute the ordered triple (x, y, z).

PART II: 10 minutes | **NYCIML Contest Three** | **Spring 2006**

S06B15. I roll three fair 6-sided dice. Compute the probability that the sum of the three numbers is 15.

S06B16. A root of the polynomial $x^3 - 5x^2 + cx - 45$ is of the form ki where $i = \sqrt{-1}$. If c and k are both positive real numbers, Compute k.

PART III: 10 minutes | **NYCIML Contest Three** | **Spring 2006**

S06B17. A parallelogram has three vertices whose coordinates are (-3, 3), (0, 0), and (3, 2). Compute all possible coordinates for the fourth vertex.

S06B18. Compute the number of ways to arrange the numbers 1, 2, 3, 4, and 5 in a line such that no two even numbers are next to one another?

ANSWERS

13.	7	15.	$\frac{5}{108}$	17.	(0, 5), (-6, 1), (6, -1)
14.	(11/3, 2/3, 2/3)	16.	3	18.	72

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SENIOR B DIVISION **CONTEST NUMBER FOUR** **SPRING 2006**

PART I: 10 minutes **NYCIML Contest Four** **Spring 2006**

S06B19. A clock runs 20 minutes slower every hour. It does not distinguish between AM and PM when telling the time. If the clock is set to the correct time at 12:00 one day, compute how many hours will it take until the clock reads the correct time again?

S06B20. Compute how many angles x, measured in radians, satisfy $\sin(x)=0$ and $0 \le x \le 2006$?

PART II: 10 minutes **NYCIML Contest Four** **Spring 2006**

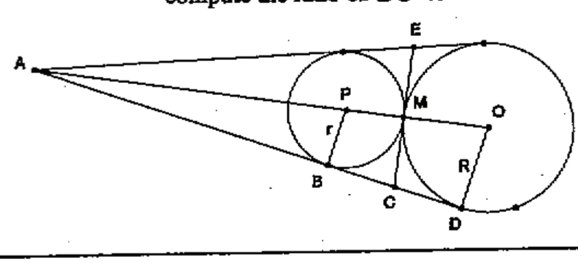
S06B21. Let $x^4+ax^3+bx^2+cx+d$ be a polynomial with integer coefficients that has $\sqrt{\sqrt{3}+1}$ as a root. Compute (a,b,c,d).

S06B22. Let points A and B be points on a circle of radius 5 such that $\overline{AB}=3$. If point C is another point on the circle, then the area of the largest ΔABC that can be inscribed in the circle will be of the form $\frac{a+b\sqrt{c}}{4}$ where a, b, and c are positive integers, and c is as small as possible. Compute (a, b, c).

PART III: 10 minutes **NYCIML Contest Four** **Spring 2006**

S06B23. If $(\log_2 9)(\log_5 4)(\log_3 25)=x$, and x is an integer, compute the value of x.

S06B24. In the following diagram, segment $\overline{AD}$ is tangent to circle P and circle O at points B and D respectively. Moreover, $\overline{CE}$ is tangent to both circles at point M. If r is the radius of circle P, and R is the radius of circle O, and if $R:r=5:3$, compute the ratio of $\overline{BC}$ to $\overline{AB}$.



ANSWERS

19.	36	21.	$(0,-2,0,-2)$	23.	8
20.	639	22.	(30, 3, 91)	24.	1/3 or 1:3

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **SPRING 2006**

PART I: 10 minutes **NYCIML Contest Five** **Spring 2006**

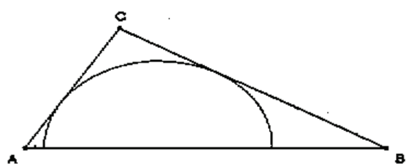
S06B25. An interior angle of a regular polygon is 170°. Compute the number of sides of the polygon.

S06B26. Let A_1, A_2, A_3, A_4, A_5 be the first five terms of an arithmetic progression with a common difference that is not zero.
Let G_1, G_2, G_3 be the first three terms of a geometric progression with common ratio m where $m \neq 1$.
If $A_1 = G_1 \neq 0$, $A_2 = G_2$, and $A_5 = G_3$, compute the value of m.

PART II: 10 minutes **NYCIML Contest Five** **Spring 2006**

S06B27. $(\sqrt{2})(\sqrt[3]{2})(\sqrt[4]{2}) = \sqrt[a]{2^b}$ and a and b are relatively prime positive integers. Compute the ordered pair (a, b).

S06B28. In ΔABC, $AB = 10$, $BC = 8$, and $AC = 6$. If a semicircle is inscribed inside the triangle with the semicircle tangent to sides AC and BC, compute the radius of the semicircle.



PART III: 10 minutes **NYCIML Contest Five** **Spring 2006**

S06B29. Express the value of the following series as a rational number in simplest form.

$$\frac{1}{5^1} - \frac{2}{5^2} + \frac{3}{5^3} + \frac{1}{5^4} - \frac{2}{5^5} + \frac{3}{5^6} + \frac{1}{5^7} - \frac{2}{5^8} + \frac{3}{5^9} + \cdots$$

S06B30. Two points A and B lie on the circumference of a circle such that the central angle created by A and B is 70°. Point C is to be randomly placed on the circumference of the circle creating ΔABC. Compute the probability that ΔABC is acute?

ANSWERS

25.	36	27.	(12, 13)	29.	9/62
26.	3	28.	24/7	30.	7/36

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER ONE** **SOLUTIONS** **Spring 2006**

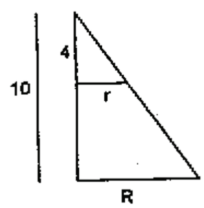
SOLUTIONS

S06B1 If we let the original number of men to women be $\frac{X}{Y}$, then we obtain $\frac{X+18}{Y}=\frac{3}{1}$ and $\frac{X}{Y-y}=\frac{3}{1}$. From these we can generate two equations: $\begin{matrix}3Y=X+18\\3Y-3y=X\end{matrix}$. Solving these yields $y = 6$. Note, X will turn out to have to be a multiple of 3 for Y to be an integer.

S06B2 If we add 2 to both sides of the equation, then we obtain $xy+2x+y+2=27$. This factors into $(x+1)(y+2)=27$. Because x and y are both positive, the only possibilities are $\begin{matrix}x+1=9\\y+2=3\end{matrix}$ and $\begin{matrix}x+1=3\\y+2=9\end{matrix}$. Thus, we get (8, 1) and (2, 7).

S06B3 Using the volume of a cone, $\frac{1}{3}\pi r^2h=\frac{1}{3}\pi r^2(4)=12\pi$, we find that $r = 3$. Now we can use similar triangles to find the radius of the larger cone.

$$\frac{r_{small}}{R_{big}}=\frac{h_{small}}{H_{big}}$$
$$\frac{3}{R}=\frac{4}{10}\Rightarrow R=\frac{15}{2}$$



S06B4 We want to find a low power of 2 that has a "nice" remainder when divided by 17. We note that 2^4 leaves a remainder of 16 (or -1), thus $2^8=256$ leaves a remainder of 1 when divided by 17 (256 = 17*15+1). Because $2^{2000}=\left(2^8\right)^{250}$, 2^{2000} leaves a remainder of 1 when divided by 17. The remainder that 2^{2005} leaves is therefore the same as the remainder $2^5=32$ leaves. Thus, the answer is 15.

S06B5 In order to minimize the area, we need to minimize the radius. We can accomplish this if the two points on the circle are the endpoints of a diameter. Therefore,

$2r=\sqrt{(8-2)^2+(-1-7)^2}=\sqrt{36+64}=10$. Area = 25π.

S06B6 If we write out the first 10 "Helen" numbers in terms of a and b, we get: a, b, $a+b$, $a+2b$, $2a+3b$, $3a+5b$, $5a+8b$, $8a+13b$, $13a+21b$, and $21a+34b$. Note that all of these coefficients are Fibonacci numbers. If we take the sum of these, we get $55a+88b=11(5a+8b)=462$. Therefore, $5a+8b=42$. Because a and b are both integers and $0<a<b$, the only possibility is $a = 2$ and $b = 4$.

S06B7 If the points are collinear, then the slopes between them must be equal. Therefore, $\frac{(k-5)-(7)}{(2k)-(k)}=\frac{(7)-(k)}{(k)-(3)}$. After simplifying and cross-multiplying, we obtain the following quadratic

$$k^2-12k-3k+36=7k-k^2$$

equation in k: $2k^2-22k+36=0$

$$2(k-9)(k-2)=0$$

$$\therefore k=9,2$$

S06B8 We need to determine how many factors of 5 there are in $(5^3)!$ There will enough factors of 2 to make "10"s. There are $5^2=25$ multiples of 5 in the list of integers $1,2,3,\cdots,5^3$. However, there are 5 multiples of 25, each of which supplies an additional factor of 5. Finally, there is one multiple of 125 which supplies still another factor of 5. Thus there are 25 + 5 + 1 = 31 factors of 10.

S06B9 We could express the expression as follows and simplify using the addition formulas and the fact that $\sin(45°)=\cos(45°)$:

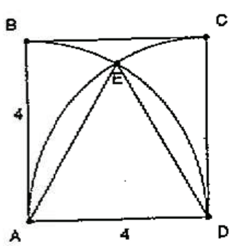
$$\frac{\sin(15°)+\cos(15°)}{\sin(75°)+\cos(75°)}=\frac{\sin(45°-30°)+\cos(45°-30°)}{\sin(45°+30°)+\cos(45°+30°)}$$

$$=\frac{\sin(45°)\cos(30°)-\cos(45°)\sin(30°)+\cos(45°)\cos(30°)+\sin(45°)\sin(30°)}{\sin(45°)\cos(30°)+\cos(45°)\sin(30°)+\cos(45°)\cos(30°)-\sin(45°)\sin(30°)}=$$

$$=\frac{\cos(30°)-\sin(30°)+\cos(30°)+\sin(30°)}{\cos(30°)+\sin(30°)+\cos(30°)-\sin(30°)}=\frac{\sqrt{3}/2+\sqrt{3}/2}{\sqrt{3}/2+\sqrt{3}/2}=1$$

However, we can arrive at the solution more quickly if we note that $\sin(15°)=\cos(75°)$ and that $\cos(15°)=\sin(75°)$.

S06B10 Constructing segments $\overline{AE}$ and $\overline{DE}$ creates equilateral ΔADE having area $\frac{\sqrt{3}}{4}(4^2)=4\sqrt{3}$. Moreover, $\angle BAE=\angle CDE=30°$. Therefore, wedge ABE and wedge CDE both have area $\frac{1}{12}\pi(4^2)=\frac{4\pi}{3}$. Therefore the area of region BEC is $16-4\sqrt{3}-\frac{8\pi}{3}$.



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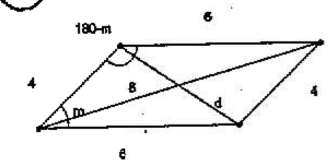

S06B11 If we convert both numbers to their base 10 representation, then we obtain $325_9=3(9^2)+2(9)+(5)=266=2b^2+2b+2=222_b$. Solving this quadratic for the positive value of b yields b = 11.

S06B12 Because consecutive angles of a parallelogram are supplementary (here m and 180°-m), the cosines of these angles are negatives of each other. By the law of cosines, we get:

$$d^2=4^2+6^2-2(4)(6)\cos(m)$$

$$8^2=4^2+6^2-2(4)(6)\cos(180-m)=4^2+6^2+2(4)(6)\cos(m)$$

Therefore, $d^2+8^2=4^2+6^2+4^2+6^2$. Solving for d, $d=2\sqrt{10}$.

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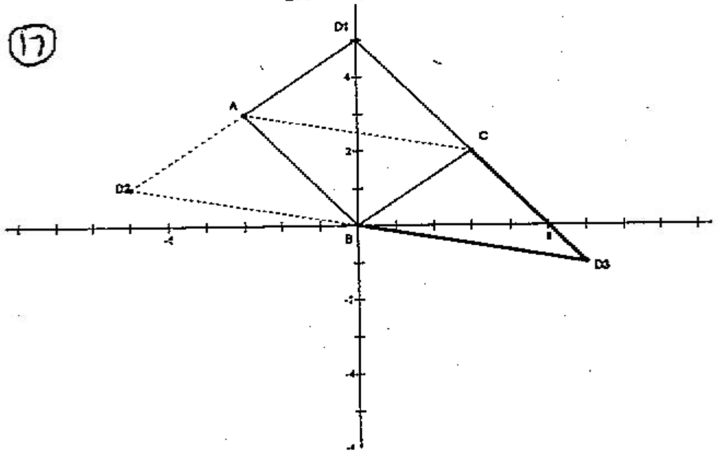
SENIOR B DIVISION **CONTEST NUMBER THREE** **SOLUTIONS** **Spring 2006**

S06B13 Let r be Jim's rate, $2r$ be Helen's rate, and $4r$ be Sigmund's rate. Working together, their rate is $7r$. Therefore, $r = \frac{1}{7}$. Working alone, it takes Jim 7 hours.

S06B14 Adding the equations, we obtain $3x+3y+3z=15$ or $x+y+z=5$. We can substitute the original expressions into the latter expression and obtain $(5-2y)+y+(1-.5y)=5$. Solving this equation, we get $y=\frac{2}{3}$. Using the first equation, $x=\frac{11}{3}$. Finally, $z=\frac{2}{3}$.

S06B15 There are 6^3 possible outcomes for rolling 3 dice. Outcomes that sum to 15 are 6-6-3 $\left(\frac{3!}{2!} = 3 \text{ ways}\right)$, 6-5-4 $(3! = 6 \text{ ways})$, and 5-5-5 (1 way). Therefore, there are a total of 10 ways to get a sum of 15. The probability is $\frac{10}{216} = \frac{5}{108}$.

(17)



S06B16 Because all of the coefficients of the polynomial are real, then any complex root must come in conjugate pairs. Thus, two of the three roots of the polynomial are $+ki$ and $-ki$. From the x^3 coefficient and the x^2 coefficient, we know the sum of the roots must be 5. $+ki$ and $-ki$ sum to 0, making the third root 5. Finally, the product of the three roots must equal 45. Thus,
$$\begin{cases} -k^2i^2 = 9 \\ k^2 = 9 \\ k = \pm 3 \\ \therefore k = 3 \end{cases}$$

S06B17 From the diagram above, we see that three parallelograms can be made with point D at (0, 5), (-6, 1), or (6, -1). We need only make sure the slopes are equal to the opposite sides.

S06B18 First we must determine how many ways there are to arrange two evens and three odds without the two evens being next to each other. There are a total of 6 ways (EOEOO, EOOEO, EOOOE, OEOEO, OEOOE, and OOEOE). Another way to get this is to start with the idea that we must have EOE and that we need to place two more odds into 3 available places—before the first E, between the two E's, and after the last E. This can be thought of as arranging EEOO. There are $\frac{4!}{2!2!} = 6$ ways to do this.

Once we have this, we need to determine how many arrangements there are for any particular arrangement of E's and O's. There are $2!3! = 12$ ways for each. Therefore, there are $(12)(6) = 72$ arrangements.

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SENIOR B DIVISION **CONTEST NUMBER FOUR** **SOLUTIONS** **Spring 2006**

S06B19 One effective way to solve the problem is to make a chart of the clock's time and the actual time. For every three actual hours that go by, only two clock hours go by. If we remember to cycle around to 0 once you get to 12 hours, we will find that the actual and clock time will be at 12 (or 0) after 36 hours.

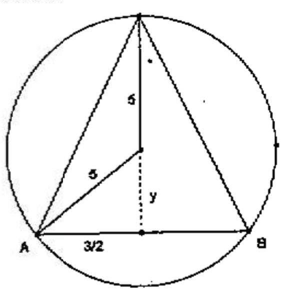
Given that the actual time and clock time each cycle through 12 hours, congruences can be used to solve the problem. If x is the actual number of hours that pass, then the clock will go $(2/3)x$. Thus we get the congruence $x \equiv \frac{2}{3}x \pmod{12}$. This simplifies to $\frac{1}{3}x \equiv 0 \pmod{12}$. Thus, $x = 0$ (the start time) works. The next value of x that works is 36.

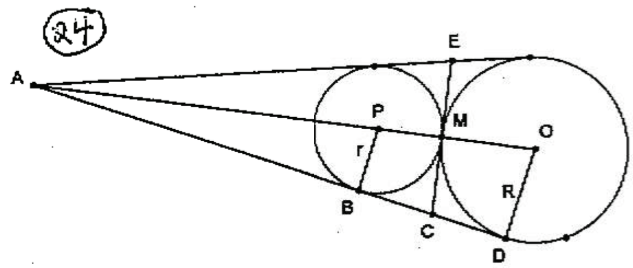
S06B20 $\sin(x) = 0$ for all $x = n\pi$ where n is an integer. Thus, we need the largest integer n such that $n\pi \le 2006$. Approximating π to 3.1416 yields $n \le 638$. Because n can also equal 0, we obtain a total of 639 values. Note, if we approximate π to 3, we obtain a value of n that is much too high.

S06B21 If $\sqrt{\sqrt{3}+1}$ is a root, then $\left(x - \sqrt{\sqrt{3}+1}\right)$ is a factor. We allow:

$$x - \sqrt{\sqrt{3}+1} = 0$$
$$x = \sqrt{\sqrt{3}+1}$$
$$x^2 = \sqrt{3}+1$$
$$\left(x^2 - 1\right) = \sqrt{3}$$
$$x^4 - 2x^2 + 1 = 3 \Rightarrow (a,b,c,d) = (0,-2,0,-2)$$

S06B22 If we let $\overline{AB}$ be the base of the triangle, we need to maximize the height to this base in order to maximize the area of the triangle. An isosceles triangle will do this.





To find the value of y: $y = \sqrt{(5)^2 - \left(3/2\right)^2} = \frac{\sqrt{91}}{2}$. The area is therefore $\frac{30}{4} + \frac{3\sqrt{91}}{4}$.

S06B23 Using the change of base rule, we can rewrite the expression as

$$\left(\frac{\log 9}{\log 2}\right)\left(\frac{\log 4}{\log 5}\right)\left(\frac{\log 25}{\log 3}\right) = \left(\frac{2\log 3}{\log 2}\right)\left(\frac{2\log 2}{\log 5}\right)\left(\frac{2\log 5}{\log 3}\right) = (2)(2)(2) = 8.$$

S06B24 The key to the problem is to realize that $\overline{BC} = \overline{CM} = \overline{CD}$ by virtue of the fact that they are tangent segments to the two circles and that point M is the point of tangency of both circles. Now we can use the fact that ΔPBA and ΔODA are similar right triangles and create the proportion:

$$\frac{R}{r} = \frac{\overline{AD}}{\overline{AB}} = \frac{\overline{AB} + \overline{BC} + \overline{CD}}{\overline{AB}} = \frac{\overline{AB} + 2\cdot\overline{BC}}{\overline{AB}} = 1 + \frac{2\cdot\overline{BC}}{\overline{AB}} = \frac{5}{3}$$ Thus, $$\frac{\overline{BC}}{\overline{AB}} = \frac{1}{3}$$

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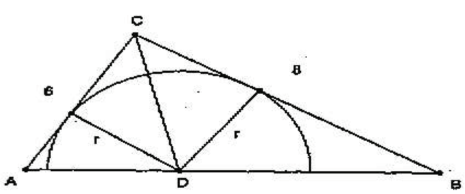
SENIOR B DIVISION **CONTEST NUMBER FIVE** **SOLUTIONS** **Spring 2006**

S04B25 An interior angle of a polygon is given by $\frac{(n-2)(180)}{n}$. Setting this equal to 170 and solving for n yields $n = 36$.

S04B26 We can express the first five terms of the arithmetic progression as: $f, f+a, f+2a, f+3a$, and $f+4a$. Likewise, the first three terms of the geometric progression f, fm, and fm^2. Equating the second terms of each progression yields $a = f(m-1)$. Setting $f + 4a = fm^2$, we can obtain $4a = f(m^2-1) = f(m-1)(m+1)$. Finally, dividing yields $4 = m + 1$, or $m = 3$.

S06B27 $(\sqrt{2})(\sqrt[3]{2})(\sqrt[4]{2}) = \left(2^{1/2}\right)\left(2^{1/3}\right)\left(2^{1/4}\right) = 2^{1/2+1/3+1/4} = 2^{13/12} = \sqrt[12]{2^{13}}$

S04B28 Let D be the center of the semi circle. If we draw radii to the tangent sides and segment CD, we see that the area of the triangle is $\frac{1}{2}(6)(r) + \frac{1}{2}(8)(r) = 7r$. Because the triangle is a right triangle, its area is $\frac{1}{2}(6)(8) = 24$. $\therefore r = \frac{24}{7}$.



S04B29 One way to compute the value of the series is to find the sum of the first three terms: $\frac{1}{5} - \frac{2}{5^2} + \frac{3}{5^3} = \frac{25-10+3}{5^3} = \frac{18}{5^3}$. We can now rewrite the series as a single geometric series as follows: $\frac{18}{(5^3)^1} + \frac{18}{(5^3)^2} + \frac{18}{(5^3)^3} + \cdots = \frac{\frac{18}{(5^3)}}{1 - \frac{1}{5^3}} = \frac{18}{124} = \frac{9}{62}$. Another method has us treating the original series as the sum of three geometric series (one with numerator 1, one with numerator 2, and one with numerator 3).

S04B30 Consider diameters BQ and AP. If point C were at either Q or P, then ΔABC would be a right triangle. Moreover, if point C lies on either arc BAQ or arc ABP, then the triangle would be obtuse. Thus, the only place C can go is on the 70° arc QP. The probability that ΔABC is acute is $\frac{70°}{360°} = \frac{7}{36}$.