New York City
Interscholastic
Mathematics
League

SENIOR B DIVISION | **CONTEST NUMBER ONE** | **FALL 2005**

PART I: 10 minutes | **NYCIML Contest One** | **Fall 2005**

F05B01 Compute the distance from the center of a circle of radius 3 inches to a chord of length 5 inches.

F05B02 The line $y = mx + m$ intersects the graph of $y = x^2$ at $x = m + 2$. Compute the value of m.

PART II: 10 minutes | **NYCIML Contest One** | **Fall 2005**

F05B03 Clio travels to and from work along the same route. She drove at a constant rate of 40 mph going to work and 50 mph coming home. What was her average rate for the entire round trip?

F05B04 In trapezoid $ABCD$, $\overline{AB}$ is parallel to $\overline{CD}$, AB=7, and CD=15. Segment $\overline{HM}$ is drawn parallel to $\overline{AB}$ with point H lying on $\overline{AD}$ and point M lying on $\overline{BC}$. If $AH : HD = 3:2$, compute the length of $\overline{HM}$.

PART III: 10 minutes | **NYCIML Contest One** | **Fall 2005**

F05B05 In the decimal expansion of the rational number $\frac{3}{7}$, what is the 2005^{th} digit after the decimal point?

F05B06 If x, y, and z are positive integers, compute the number of solutions (x, y, z) that solve the equation $x + y + z = 21$.

ANSWERS

1. $\sqrt{11}/2$
2. -4
3. $400/9$ or $44\frac{4}{9}$
4. $59/5$ or $11\frac{4}{5}$
5. 4
6. 190

New York City Interscholastic Mathematics League

SENIOR B DIVISION | **CONTEST NUMBER TWO** | **FALL 2005**

PART I: 10 minutes | **NYCIML Contest Two** | **Fall 2005**

F05B07 A bag contains only red and gold marbles. The probability of selecting a red marble is $\frac{2}{5}$. If 20 red marbles are added to the bag, the probability of selecting a red marble is now $\frac{4}{7}$. Compute the number of <u>gold</u> marbles in the bag.

F05B08 What is the total number of positive integral factors of $(60)^5$?

PART II: 10 minutes | **NYCIML Contest Two** | **Fall 2005**

F05B09 If $\lfloor x \rfloor$ represents the greatest integer less than or equal to x, and if $|x|\lfloor x \rfloor = -\frac{10}{3}$, compute the value of x.

F05B10 If x and y are positive integers, find all (x, y) that solve $x^2 - y^2 = 275$.

PART III: 10 minutes | **NYCIML Contest Two** | **Fall 2005**

F05B11 Let p and q be the roots of $3x^2 - 10x + 4 = 0$. Compute the value of $\frac{1}{p} + \frac{1}{q}$.

F05B12 Two circles whose centers are 10 cm apart have a common external tangent segment of length 8 cm and a common internal tangent segment of length $\sqrt{34}$. Compute the value of the product of the two radii.

<u>ANSWERS</u>

7. 30
8. 396
9. $-5/3$ or $-1\frac{2}{3}$
10. (138, 137), (30, 25), and (18, 7)
11. $5/2$ or $2\frac{1}{2}$
12. $15/2$ or $7\frac{1}{2}$

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER THREE** **FALL 2005**

PART I: 10 minutes **NYCIML Contest Three** **Fall 2005**

F05B13 Triangle ABC is isosceles with AB=8 and BC=2. Find the area of the triangle.

F05B14 Compute the area of the region enclosed by the intersection of the graphs of $y \le -|x+1|+4$ and $y \ge |x+1|$.

PART II: 10 minutes **NYCIML Contest Three** **Fall 2005**

F05B15 What is the sum of the coefficients in the expansion of $(3-4x)^5$?

F05B16 Compute the value of $\cos(Arc\cos\frac{3}{5}+Arc\sin\frac{3}{5})$.

PART III: 10 minutes **NYCIML Contest Three** **Fall 2005**

F05B17 If $\log(x+1)+\log(x+2)=\log(2x+22)$, solve for x.

F05B18 A regular hexagon is inscribed inside of a circle. The circle is inscribed inside a larger regular hexagon. Compute the ratio of the area of the larger hexagon to the area of the smaller hexagon.

ANSWERS

13. $3\sqrt{7}$
14. 8
15. -1
16. 0
17. 4
18. $\frac{4}{3}$

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SENIOR B DIVISION | **CONTEST NUMBER FOUR** | **FALL 2005**

PART I: 10 minutes | **NYCIML Contest Four** | **Fall 2005**

F05B19 If x-1, $2x$+3, and $5x$-1 are the first three terms of an arithmetic progression, compute the value of x.

F05B20 Find the smallest integer greater than 1 that leaves a remainder of 1 when divided by 3, 4, 5, 6, 7, and 8.

PART II: 10 minutes | **NYCIML Contest Four** | **Fall 2005**

F05B21 In triangle ABC, AB=10, BC=24, and AC=26. If the triangle is inscribed inside a circle, what is the area of this circle?

F05B22 Solve for all real values of x: $9^x - 3^{x+1} = 54.$

PART III: 10 minutes | **NYCIML Contest Four** | **Fall 2005**

F05B23 If $\dfrac{(x!)!}{x!} = 120$, compute the value of x.

F05B24 Helen and Jim take turns tossing a fair coin. Helen flips first. Whoever tosses a "heads" first wins the game. If the probability that Jim wins can be expressed as the fraction $\frac{a}{b}$ where a and b are relatively prime integers, compute (a, b).

ANSWERS

19. 4
20. 841
21. 169π
22. 2
23. 3
24. (1, 3)

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **FALL 2005**

PART I: 10 minutes **NYCIML Contest Five** **Fall 2005**

F05B25 If $(2^4)(4^8)(8^{16})(16^{32}) = 32^x$, compute the value of x.

F05B26 A rectangular solid has a face with area 24 in^2, a face with area 48 in^2, and a face with area 32 in^2. If the volume of the rectangular solid is V in^3, where V is an integer, compute the value of V.

PART II: 10 minutes **NYCIML Contest Five** **Fall 2005**

F05B27 Eight points lie in a plane such that no three points are collinear. Compute the number of distinct triangles that can be made in which the vertices of each triangle are three of the eight points.

F05B28 Find all real values of x for which $\left(\log x^2\right)^2 + \log x^2 = 2$ where the base of the logarithm is 10.

PART III: 10 minutes **NYCIML Contest Five** **Fall 2005**

F05B29 If 747,A44,31B is a 9-digit base 10 number which is divisible by 15, compute all ordered pairs (A, B).

F05B30 In quadrilateral ILBK, IL=7, BK=7, IK=6, and diagonal LK=5. If $m\angle LIK = m\angle LKB$, compute the length of LB in simplest radical form.

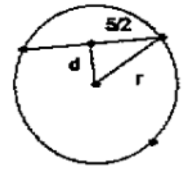
ANSWERS

25. $196/5$ or $39\frac{1}{5}$
26. 192
27. 56
28. 1/10, $\sqrt{10}$
29. (0, 0), (3, 0), (6,0), (9, 0), (1, 5), (4, 5), & (7, 5)
30. $2\sqrt{6}$

NYCIML SENIOR B DIVISION CONTEST NUMBER ONE FALL 2005

Solutions for Contest 1

F05B01 Consider the diagram below. The distance d from the center to the chord is the perpendicular distance to the chord. Moreover, this segment bisects the chord. Thus,



$$d^2 + \left(\tfrac{5}{2}\right)^2 = r^2$$

$$d = \sqrt{9 - \tfrac{25}{4}} = \frac{\sqrt{11}}{2}.$$

F05B02 By substitution, $mx + m = x^2$ becomes

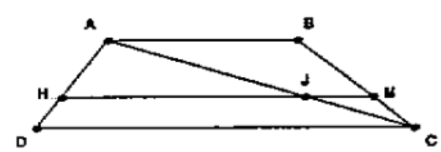
$$m(m+2) + m = (m+2)^2$$
$$m^2 + 3m = m^2 + 4m + 4$$
$$m = -4$$

F05B03 If D is the distance to work, then the entire trip has distance $2D$. The time to work is given by $t_1 = \dfrac{D}{40}$ while the time returning home is $t_2 = \dfrac{D}{50}$. Thus,

$$2D = (t_1 + t_2)R$$
$$2D = \left(\frac{D}{40} + \frac{D}{50}\right)R$$
$$R = \frac{2}{\frac{1}{40} + \frac{1}{50}} = \frac{400}{9}.$$

Note: This is the *harmonic mean* of the two rates.

F05B04 One method of solution is to construct diagonal $\overline{AC}$ and let J be the point of intersection of $\overline{AC}$ and $\overline{HM}$. Now we have $\Delta AHJ \sim \Delta ADC$ and $\Delta CMJ \sim \Delta CBA$. Therefore, $\dfrac{\overline{HJ}}{\overline{DC}} = \dfrac{3}{5}$ and $\dfrac{\overline{JM}}{\overline{AB}} = \dfrac{2}{5}$. Substituting the lengths for $\overline{DC}$ and $\overline{AB}$, we solve and get $\overline{HJ} = 9$ and $\overline{JM} = \dfrac{14}{5}$. Therefore, $\overline{HM} = \dfrac{59}{5}$.



F05B05 The decimal expansion of $3/7 = 0.428571428571\ldots$ repeats every six places. Since $2005 = 6(334) + 1$, the 2005^{th} digit is the same as the first digit, which is 4.

F05B06 Because x, y, and z are positive integers, we can imagine a series of 21 "1's" lined up as follows: 1 1 1 1 ... 1 | 1 1 ... 1 1 1 | 1 1 ... 1 1. We need to separate the "1's" into three groups by placing 2 dividers between the "1's". The leftmost group will be the value of x, the middle group will be the value of y, and the rightmost group the value of z. If we do not allow two dividers to occupy the same gap between the "1's", we ensure that x, y, and z will be positive. There are 20 gaps and we need to place 2 dividers. Thus, there are ${}_{20}C_2 = \dfrac{20 \cdot 19}{2} = 190$

Solutions to Contest 2

F05B07 If x represents the number of gold marbles and T represents the original total number of marbles, then $P(gold) = \frac{3}{5} = \frac{x}{T}$ originally and $P(gold) = \frac{3}{7} = \frac{x}{(T+20)}$.

Solving the system of equations $\begin{matrix} 5x = 3T \\ 7x = 3T + 60 \end{matrix}$, we get $x = 30$.

F05B08 In general, if the prime factorization of a number is $p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$ where $p_1, p_2, \ldots, p_n$ are distinct primes and $e_1, e_2, \ldots, e_n$ are positive integers, the number of positive integral factors is $(e_1+1)(e_2+1)\cdots(e_n+1)$. Here, $(60)^5 = (2^2 \cdot 3 \cdot 5)^5 = (2^{10} \cdot 3^5 \cdot 5^5)$. Thus, the number of factors is $(10+1)(5+1)(5+1) = 396$.

F05B09 $|x|$ is always positive; therefore, $\lfloor x \rfloor$ is negative, thereby forcing x to be negative. Testing values for x yields $-1 < x < -2$. Therefore, $\lfloor x \rfloor = -2$, making $x = -\frac{5}{3}$. Note: You could also examine the inequality $|x|\lfloor x \rfloor \le -x^2$, for $x<0$.

F05B10 Factoring the expression yields $(x+y)(x-y) = 275$. Because x and y are both positive integers, $x+y > x-y$, and $x > y$. Moreover, $(x+y)$ and $(x-y)$ must be factors of 275. These factors are 275 & 1, 5 & 55, and 11 & 25. Solving the systems of equations in each case yields the solutions (138, 137), (30, 25), and (18, 7).

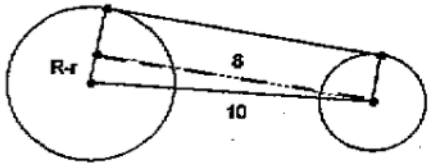
F05B11 Because the roots of a quadratic solves the equation $ax^2 + bx + c = 0$, or alternatively $x^2 + \frac{b}{a}x + \frac{c}{a} = (x-p)(x-q) = 0$, we know that $p + q = -\frac{b}{a}$ and $pq = \frac{c}{a}$.

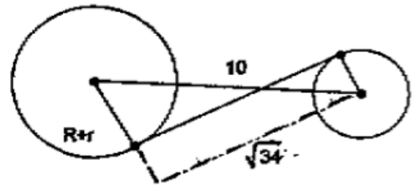
Thus, $\frac{1}{p} + \frac{1}{q} = \frac{p+q}{pq} = \frac{10/3}{4/3} = \frac{5}{2}$.

F05B12 Let R and r be the radii of the circles. Consider the diagrams below. For the external tangent, we get a right triangle with hypotenuse 10 and legs 8 and $(R-r)$. For the internal tangent, we get a right triangle with hypotenuse 10 and legs $\sqrt{34}$ and $(R+r)$.

This produces the system of equations: $\begin{matrix} (R-r)^2 + 8^2 = 10^2 \\ (R+r)^2 + \sqrt{34}^2 = 10^2 \end{matrix}$.

Expanding: $\begin{matrix} R^2 - 2Rr + r^2 + 64 = 100 \\ R^2 + 2Rr + r^2 + 34 = 100 \end{matrix}$. Subtracting yields $4Rr = 30$ $\therefore Rr = \frac{15}{2}$.





Solutions to Contest 3

F05B13 Because the triangle is isosceles there are only two possibilities for the side lengths: 8-8-2 or 8-2-2. The latter violates the triangle inequality. Therefore, the sides are 8-8-2. Therefore, altitude is given by $8^2 = h^2 + 1^2$. Thus, $h = \sqrt{63}$.
The area is then given by $A = \frac{1}{2}bh = \frac{1}{2}(2)\left(\sqrt{63}\right) = \sqrt{63} = 3\sqrt{7}$. Note: You could also use Heron's formula.

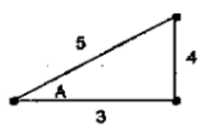
F05B14 The points of intersection of the two graphs are at $x=-3$ and $x=1$. Moreover, the region is a square. Thus, the diagonal of the square is 4, and the area is $\frac{1}{2}d^2 = \frac{1}{2}\cdot 4^2 = 8$.

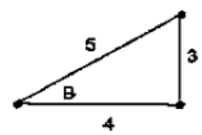
F05B15 Rather than expand the expression, set $x = 1$ yielding $(3 - 4(1))^5 = -1$.

F05B16 Let $A = Arc\cos \frac{3}{5}$ and $B = Arc\sin \frac{3}{5}$.

Using the cosine addition formula:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B = \left(\tfrac{3}{5}\right)\left(\tfrac{4}{5}\right) - \left(\tfrac{4}{5}\right)\left(\tfrac{3}{5}\right) = 0$$





Note also, that A and B are complementary, thus $\cos(A+B)=\cos(90°)=0$

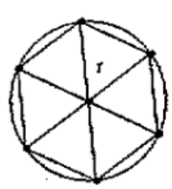
F05B17 Applying the rule that $\log A + \log B = \log AB$, we obtain $\log((x+1)(x+2)) = \log(2x+22)$. Thus, $\log(x^2+3x+2) = \log(2x+22)$.

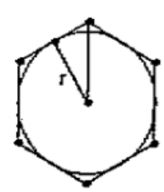
We can remove the logs and solve as follows:

$$\begin{cases} x^2 + 3x + 2 = 2x + 22 \\ x^2 + x - 20 = 0. \\ x = 4, -5. \end{cases}$$

If we test the two values in the original expression, we must reject $x = -5$. Therefore, the only solution is $x = 4$.

F05B18 A regular hexagon is composed of 6 equilateral triangles. Each triangle has an area of $\frac{\sqrt{3}}{4}r^2$; therefore, the hexagon has an area of $\frac{3\sqrt{3}}{2}r^2$. The larger hexagon will be composed of 6 equilateral triangles of area $\frac{\sqrt{3}}{4}\left(\frac{2r}{\sqrt{3}}\right)^2$. Thus the larger hexagon has area $2\sqrt{3}r^2$. Thus, the ratio of the larger hexagon to the smaller hexagon is $\frac{4}{3}$.





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NYCIML SENIOR B DIVISION CONTEST NUMBER FOUR FALL 2005

Solutions to Contest 4

F05B19 In general, if A, B, and C form an arithmetic progression, then $B = \frac{A+C}{2}$, or $2B = A + C$. Therefore,

$$2(2x+3) = (x-1) + (5x-1)$$
$$4x + 6 = 6x - 2$$
$$\therefore x = 4.$$

F05B20 If x is the integer, then we get a remainder of 1 when we divide x by 3, 4, 5, 6, 7, or 8, then $x-1$ must be a multiple of 3, 4, 5, 6, 7, and 8. In order to minimize $x-1$, we need to find the least common multiple of 3, 4, 5, 6, 7, and 8. This is 840. Therefore, $x = 841$.

F05B21 ΔABC is a right triangle; therefore, if you inscribe the triangle in a circle, then the hypotenuse is the diameter of the circle. Therefore, the radius is 13, and the area is 169π.

F05B22 We can rewrite the expression in terms of powers of 3 as follows: $\left(3^2\right)^x - 3^1 3^x - 54 = 0$, or simply $\left(3^x\right)^2 - 3 \cdot 3^x - 54 = 0$.

If we let $y = 3^x$, then we have the quadratic $y^2 - 3y - 54 = 0$. This has solutions $y = 9$ and $y = -6$. Substituting back for y, we can reject the root -6. Therefore, $3^x = 9$, and $x = 2$.

F05B23 We can consider the expression $\frac{(x!)!}{x!}$ as $\frac{(x!)(x!-1)(x!-2)\cdots(3)(2)(1)}{x!}$. Dividing by $x!$ yields $(x!-1)(x!-2)\cdots(3)(2)(1) = (x!-1)!$ We know now that $(x!-1)! = 120$, forcing $x!-1 = 5$, or simply $x! = 6$. Thus, $x=3$.

F05B24 If we consider the first opportunity that Jim could win, Helen must flip a "tails", and Jim must flip a "heads". The probability of this occurring is $P(tails) \cdot P(heads) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. This however is not the only way Jim could win. Helen could flip "tails," then Jim flips "tails," then Helen flips "tails," and finally Jim flips "heads". This probability is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$. If we continue this process to find the next opportunity for Jim, we will discover that the probabilities follow a geometric progression whose common ratio is $\frac{1}{4}$. We need to find the sum of this geometric series. Thus, the probability that Jim wins is

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots + \frac{1}{4^n} + \cdots = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$ Thus, (1, 3).

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NYCIML SENIOR B DIVISION CONTEST NUMBER FIVE FALL 2005

Solutions for contest 5

F05B25 Converting the expression to powers of 2 yields

$$\begin{cases} 2^4\cdot\left(2^2\right)^8\cdot\left(2^3\right)^{16}\cdot\left(2^4\right)^{32}=\left(2^5\right)^x \\ 2^4\cdot 2^{16}\cdot 2^{48}\cdot 2^{128}=2^{5x} \\ 2^{196}=2^{5x} \\ \therefore x=\dfrac{196}{5}. \end{cases}$$

F05B26 If we let a=length, b=width, and c=height, then the area yield the following: ab=24, ac=48, and bc=32. Multiplying the three expressions together yields the following:

$$a^2b^2c^2 = 24\cdot 48\cdot 32 = 2^3\cdot 3\cdot 2^4\cdot 3\cdot 2^5$$

$$(abc)^2 = 2^{12}\cdot 3^2$$

The quantity abc is the volume of the rectangular solid; therefore,

$abc = \sqrt{2^{12}\cdot 3^2} = 2^6\cdot 3 = 192$. Note: do not mark incorrect if units are missing.

F05B27 We are selecting 3 points from a group of 8 to be the vertices of the triangle. Because the order that we select the points is not important, we use ${}_8C_3 = \dfrac{8!}{3!5!} = 56$.

F05B28 Let $y = \log x^2$. Then $y^2 + y - 2 = (y+2)(y-1) = 0$ and $y = -2, 1$.
Then $y = \log x^2 = 2\log x = -2, 1$, and $x = 1/10, \sqrt{10}$

F05B29 If the number is divisible by 15, then it must be divisible by both 5 and 3. For divisibility by 5, the unit's digit needs to be 0 or 5. For divisibility by 3, the sum of the digits of the number must be a multiple of 3. The sum of the digits is $30+A+B$. If B=0, then A could be 0, 3, 6, or 9. If B=5, then A could be 1, 4, or 7.
Therefore, all possible (A,B) are (0,0), (3, 0), (6, 0), (9, 0), (1, 5), (4, 5), & (7, 5).

F05B30 If we let $\theta = m\angle LIK = m\angle LKB$, we can use the law of cosines to solve for $\cos(\theta)$. Thus, $7^2+6^2-2\cdot 7\cdot 6\cdot\cos(\theta)=5^2$, or $\cos(\theta)=\dfrac{5}{7}$. We can now use the law of cosines again to find LB.

$$\begin{cases} 5^2+7^2-2\cdot 5\cdot 7\cos(\theta)=(LB)^2 \\ 25+49-(70)\left(\dfrac{5}{7}\right)=(LB)^2 \\ 24=(LB)^2 \\ \therefore LB = 2\sqrt{6}. \end{cases}$$