New York City
Interscholastic
Mathematics
League

JUNIOR DIVISION **CONTEST NUMBER ONE** **FALL 2005**
PART I: 10 minutes **NYCIML Contest One** **Fall 2005**

F05J1. If $720x$ is the cube of a positive integer, compute the smallest possible x.

F05J2. Two concentric circles are drawn such that a chord of the larger circle is tangent to the smaller circle. If the length of the chord is 12, compute the area of the region between the two circles.

PART II: 10 minutes **NYCIML Contest One** **Fall 2005**

F05J3. Compute the number of positive integral factors of 1002001.

F05J4. P is an interior point in rectangle $ABCD$. $PA = 6$, $PC = 5$, and $PB = 7$. Compute PD.

PART III: 10 minutes **NYCIML Contest One** **Fall 2005**

F05J5. If $x+\frac{1}{x}=5$, compute $x^3+\frac{1}{x^3}$.

F05J6. If four fair dice are rolled, compute the probability that the sum is 10.

ANSWERS:

F05J1.	**300**
F05J2.	**36π**
F05J3.	**27**
F05J4.	**$2\sqrt{3}$**
F05J5.	**110**
F05J6.	**$\frac{5}{81}$**

New York City Interscholastic Mathematics League

JUNIOR DIVISION	CONTEST NUMBER TWO	FALL 2005
PART I: 10 minutes	NYCIML Contest Two	Fall 2005

F05J7. Compute the length of the shortest altitude of a triangle whose sides measure 5, 12, and 13.

F05J8. Compute the largest possible value of n such that 3^n is a factor of 100!

PART II: 10 minutes **NYCIML Contest Two** **Fall 2005**

F05J9. Working alone, Arthur can paint a room in 6 hours. Working alone, Boris can paint the same room in 8 hours. If they begin the job together, and Boris takes an hour off for lunch, compute the amount of time it will take for the room to be painted.

F05J10. The measures of two sides of a triangle are 12 and 14 and the medians drawn to these sides are perpendicular to each other. Compute the length of the third side of the triangle.

PART III: 10 minutes **NYCIML Contest Two** **Fall 2005**

F05J11. The number $3{,}956{,}427{,}8ab$ is divisible by 72. Compute the ordered pair (a,b).

F05J12. Compute the sum of the digits of the first 200 even natural numbers.

ANSWERS:

F05J7.	$\frac{60}{13}$
F05J8.	**48**
F05J9.	$\frac{27}{7}$
F05J10.	$2\sqrt{17}$
F05J11.	$(6,4)$
F05J12.	**2004**

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JUNIOR DIVISION **CONTEST NUMBER THREE** **FALL 2005**

PART I: 10 minutes **NYCIML Contest Three** **Fall 2005**

F05J13. In a triangle, the measure of whose sides are 8, 15, and 17, a circle is inscribed. Compute the measure of the radius of the circle.

F05J14. Compute the number of the first 1000 natural numbers that are not divisible by either 2 or 3.

PART II: 10 minutes **NYCIML Contest Three** **Fall 2005**

F05J15. In a game of softball, the probability of Archie getting a hit in an at-bat is $\frac{1}{4}$. The probability of Betty getting a hit in an at-bat is $\frac{1}{3}$. The probability of Veronica getting a hit in an at-bat is $\frac{1}{2}$. If each of the three goes to bat once in the first inning, compute the probability that exactly one gets a hit.

F05J16. In right triangle ABC, $AC = 15$, and C is the right angle. If all of the measures of the sides of the triangle are integers, compute all possible ordered pairs (BC, AB).

PART III: 10 minutes **NYCIML Contest Three** **Fall 2005**

F05J17. Consider all positive integers with a left-most digit of 2. They are written next to each other, in ascending order, to form a number (2202122…). Compute the 2005th digit of this number.

F05J18. Compute, to the nearest second, the first time after 3:00 AM that the hands of a clock are perpendicular to each other.

ANSWERS:

F05J13.	**3**
F05J14.	**333**
F05J15.	$\frac{11}{24}$
F05J16.	**(112, 113), (20, 25), (36, 39), (8, 17)**
F05J17.	**0**
F05J18.	**3:32:44**

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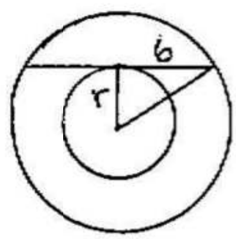
JUNIOR DIVISION **CONTEST NUMBER ONE SOLUTIONS** **FALL 2005**

F05J1. $720 = 3^2 \cdot 2^4 \cdot 5$. The exponents must be multiples of 3 in order for the number to be a perfect cube. Thus we need $3^1 \cdot 2^2 \cdot 5^2 = \mathbf{300}$.

F05J2. Let the radius of the smaller and larger circles be r and R, respectively. $\pi R^2 - \pi r^2 = \pi\left(R^2 - r^2\right) = \mathbf{36\pi}$.



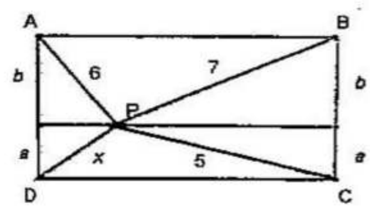
F05J3. $1{,}002{,}001 = (1001)^2 = 7^2 \cdot 11^2 \cdot 13^2$. The number of factors is $3 \cdot 3 \cdot 3 = \mathbf{27}$.

F05J4. Through P, draw a line parallel to the base.

$49 - b^2 = 25 - a^2$

$36 - b^2 = x^2 - a^2$

$13 = 25 - x^2 \rightarrow x^2 = 12 \rightarrow x = \mathbf{2\sqrt{3}}$.



F05J5.

$$x + \frac{1}{x} = 5 \rightarrow \left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2} = 25 \rightarrow x^2 + \frac{1}{x^2} = 23$$

$$\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) = 5 \cdot 23 \rightarrow x^3 + \frac{1}{x} + x + \frac{1}{x^3} = 115$$

$$x^3 + \frac{1}{x^3} = \mathbf{110}.$$

F05J6. Consider two pairs of dice and the ways that the sum can be 10. We could have $2 + 8$, $3 + 7$, $4 + 6$, $5 + 5$, $6 + 4$, $7 + 3$, and $8 + 2$. Using the known probabilities for the possible sums on a pair of dice, we get:

$$\frac{1}{36}\cdot\frac{5}{36}+\frac{2}{36}\cdot\frac{6}{36}+\frac{3}{36}\cdot\frac{5}{36}+\frac{4}{36}\cdot\frac{4}{36}+\frac{5}{36}\cdot\frac{3}{36}+\frac{6}{36}\cdot\frac{2}{36}+\frac{5}{36}\cdot\frac{1}{36}=\frac{80}{1296}=\mathbf{\frac{5}{81}}.$$

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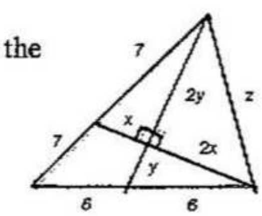
JUNIOR DIVISION **CONTEST NUMBER TWO SOLUTIONS** **FALL 2005**

F05J7. The area of this right triangle is $\frac{1}{2}\cdot 5\cdot 12 = 30 = \frac{1}{2}\cdot 13\cdot h \rightarrow h = \mathbf{\frac{60}{13}}$.

F05J8. There are 33 multiples of 3 which contribute at least one 3. There are 11 multiples of 9 which contribute a second 3, 3 multiples of 27 which contribute a third 3, and 1 multiple of 81, which contributes a fourth 3. $33+11+3+1=\mathbf{48}$.

F05J9. Let x be the time it takes them to paint the room working together. Arthur will paint $\frac{x}{6}$ of the room; Boris will paint $\frac{x-1}{8}$ of the room.
$\frac{x}{6} + \frac{x-1}{8} = 1$. $7x = 27 \rightarrow x = \mathbf{\frac{27}{7}}$.

F05J10. The medians of a triangle intersect 2/3 of the way from a vertex to the opposite side.



$$(2x)^2 + y^2 = 36, \text{ and } x^2 + (2y)^2 = 49 \rightarrow 5x^2 + 5y^2 = 85 \rightarrow x^2 + y^2 = 17.$$
$$z^2 = 4x^2 + 4y^2 = 68 \rightarrow z = \mathbf{2\sqrt{17}}.$$

F05J11. The number must be divisible by both 8 and 9. To be divisible by 8, ab must be 00, 08, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, or 96. In order for it to be divisible by 9, the sum of the digits must be divisible by 9. Thus, $44 + a + b = 45$ or $54 \rightarrow a + b = 1$ or 10. the only one that fits both criteria is **(6, 4)**.

F05J12. In the units column, there are 40 instances of each even integer. 40(0 +2 +4 +6 +8) = 800. In the tens column, there are 20 instances of each integer. 20(0 +1 + … +9) = 900. In the hundreds column, there are 50(1 + 2 + 3) and one 4. 900 +800 +304 = **2004**.

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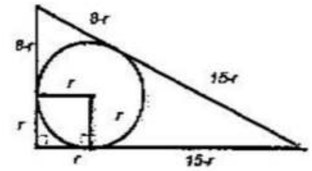
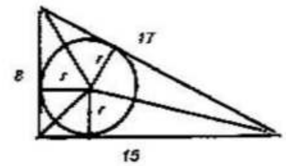
JUNIOR DIVISION **CONTEST NUMBER THREE SOLUTIONS** **FALL 2005**





F05J13. It is a right triangle and $8-r+15-r=17\to r=\mathbf{3}$.
Alternatively:
$A=\frac{1}{2}\cdot 8\cdot 15=60=\frac{1}{2}r(8+15+17)\to r=3$.

F05J14. There are 500 multiples of 2 and 333 multiples of 3 which must be subtracted. We have, however, counted the 166 multiples of 6 twice. $1000-500-333+166=\mathbf{333}$. Or:
For every 6 integers, 2 integers of the form $6n+1$ and $6n-1$ will work. That gives us 332 integers that work up to 996. We then must add 997 for a total of 333.

F05J15. The probability is: $\frac{1}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}+\frac{3}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}+\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}=\mathbf{\frac{11}{24}}$.

F05J16. $c^2-a^2=(c+a)(c-a)=15^2=225$. There are 5 pairs of positive integers whose product is 225: 225 and 1, 75 and 3, 45 and 5, 15 and 15, and 25 and 9. Setting up simultaneous equations for each combination, we get: **(112, 113), (36, 39), (20, 25), and (8, 17)**. (15 and 15 leads to an impossible solution.)

F05J17. Looking at 220212223…29200201202…299, we see that we have accounted for $1+20+300=321$ digits. We now must account for the remaining $2005-321=1684$ digits. Since the four-digit integers beginning with 2 start at 2000, and $\frac{1684}{4}=421$, the 421st such number is 2420, and the last digit of the number is **0**.

F05J18. When the hour hand has moved x degrees, the minute hand has moved $12x$ degrees. $12x=90+90+x\to x=\frac{180}{11}$. Now, since 1 hour moves 30 degrees,

$\frac{\frac{180}{11}}{30}hrs.=\frac{180}{330}hrs.=\frac{6}{11}hrs.=32\frac{8}{11}$ minutes. To the nearest second, the time is **3:32:44 AM**.