New York City Interscholastic Mathematics League

SOPH FROSH DIVISION **CONTEST NUMBER ONE** **SPRING 2005**

PART I: 10 minutes **NYCIML Contest One** **Spring 2005**

S05SF1. At formal conferences of the United States Supreme Court, each of the nine justices shakes hands with each of the others at the beginning of the session. Compute the number of handshakes exchanged.

S05SF2. In the game of chess, two rooks are considered to be attacking each other if they were placed in the same column or the same row, but not both because each rook must occupy its own square. Compute the probability that if two rooks were randomly placed on an 8 x 8 checkerboard, they would be attacking each other.

PART II: 10 minutes **NYCIML Contest One** **Spring 2005**

S05SF3. Jeanne can mow a lawn in four hours, and Karen can mow a lawn in six hours. One day, Jeanne starts to mow a lawn, and after an hour, Karen arrives and they complete the work together. Compute the number of hours Jeanne mowed.

S05SF4. A number *abcde* consists of all of the digits from 1 to 5 inclusive. *abc* is divisible by 25, *bcd* is divisible by 2, and *cde* is prime. Compute the number *abcde*.

PART III: 10 minutes **NYCIML Contest One** **Spring 2005**

S05SF5. Compute the smallest positive integer x such that $120x$ is a perfect cube.

S05SF6. Streets in the town of Jayeville are numbered consecutively beginning with 1st Street. Avenues in the town of Jayeville are numbered consecutively beginning with 1st Avenue. Jayeville is built on a Cartesian coordinate plane and all streets go north and south and all avenues go east and west. If Danny starts at 1st Street and 1st Avenue and wants to visit Jen at 4th Street and 5th Avenue, compute how many different routes he may take if he only travels north and east.

ANSWERS:

S05SF1. **36**

S05SF2. $\mathbf{\frac{2}{9}}$

S05SF3. $\mathbf{\frac{14}{5}}$ *or* $2\frac{4}{5}$

S05SF4. **32541**

S05SF5. **225**

S05SF6. **35**

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SOPH FROSH DIVISION **CONTEST NUMBER TWO** **SPRING 2005**
PART I: 10 minutes **NYCIML Contest Two** **Spring 2005**

S05SF7. a, b and c are real numbers and $(x-2)(x^2+ax+b)=(x^2-3x+2)(x-c)$ is an identity for all real values of x. Compute the value of $a+b$.

S05SF8. Compute the total number of rectangles in a 5 row by 3 column checkerboard.

PART II: 10 minutes **NYCIML Contest Two** **Spring 2005**

S05SF9. Compute $\dfrac{a^{-1}+b^{-1}}{a+b}$ if $a=0.4$ and $b=0.25$.

S05SF10. A circle with center A has radius 3. A line segment tangent to the circle at point B is drawn from a point C outside the circle. If the area of the region of the circle that is enclosed in ΔABC is $3\pi/2$, compute the area of ΔABC.

PART III: 10 minutes **NYCIML Contest Two** **Spring 2005**

S05SF11. Bernadette was asked how many children she had. She responded: "The number of children I have is ¾ of the number of children in my family plus ¾ of a child." Compute how many children she has.

S05SF12. Compute all natural numbers which are eleven times the sum of their digits.

ANSWERS:
S05SF7. **-1**
S05SF8. **90**
S05SF9. **10**
S05SF10. $\dfrac{9\sqrt{3}}{2}$
S05SF11. **3**
S05SF12. **198**

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SOPH FROSH DIVISION **CONTEST NUMBER THREE** **SPRING 2005**

PART I: 10 minutes **NYCIML Contest Three** **Spring 2005**

S05SF13. Luis traveled from point New City to Oldtown. Luis got a ride in a car for half of the trip. The car went ten times as fast as Luis could have walked. For the second half of the trip, Luis had to take the bus, and traveled only twice as fast as he could have walked. Compute how many times as fast he completed the trip, than if he had walked the whole way.

S05SF14. The altitude to the base of an isosceles triangle is 6 and the perimeter is 36. Compute the area of the isosceles triangle.

PART II: 10 minutes **NYCIML Contest Three** **Spring 2005**

S05SF15. The average of three numbers is x. One of three numbers is y. Express in terms of x and y in simplest form the sum of the other two numbers.

S05SF16. You are given four prime numbers such that the sum of any three of them is also a prime number. Compute the smallest possible sum for the four prime numbers given.

PART III: 10 minutes **NYCIML Contest Three** **Spring 2005**

S05SF17. The perimeter of a square is increased by 1000%. Compute the percentage by which the area is increased.

S05SF18. Al's address is a three digit integer, containing no zeros. If Al rearranges the digits of his address, the new numbers he gets are all smaller. Bob's address is a three digit integer. It contains the same digits as Al's address, but in a different order. If Bob rearranges his digits, the only larger number he can get is Al's address. The sum of Al's address and Bob's address is 1233. What is Al's address?

ANSWERS:

S05SF13. $\frac{10}{3}$

S05SF14. **48**

S05SF15. $3x - y$

S05SF16. **48**

S05SF17. **12000%**

S05SF18. **621**

SOPH FROSH DIVISION **CONTEST NUMBER ONE** **SPRING 2005**

SOLUTIONS

S05SF1. Since every justice has to shake hands with every other justice, there are a total of $9 \cdot 8 = 72$ handshakes. However, it is important to note that two justices participate in each handshake, and hence we are doubling counting each handshake. Therefore, the correct answer is 72 / 2 or **36**.

S05SF2. Place one rook on the chessboard. In its column, it attacks 7 squares, and on its row, it also attacks 77 spaces. In all, it attacks a total of 14 spaces. For the second rook, there are 64 -1 = 63 squares left to place the rook. The desired probability is thus 14/ 63 = **2 / 9**.

S05SF3. By the end of the first hour, Jeanne has already completed ¼ of the yard. The two of them can mow 5/12 of a lawn in an hour. Since ¾ of the yard still needs to be completed, they together have to work for 9/5 more hours. Therefore Jeanne alone worked for **14/5** hours.

S05SF4. If *abc* is divisible by 25, we know that *bc* must = 25. If 25*d* is divisible by 2, then *d* must = 4. If 54*e* is prime then *e* must = 1. Thus the number is 32541.

S05SF5. Factoring 120*x* we get $2^3 \times 3^1 \times 5^1 \times x$, so to make each power a multiple of 3 we must have *x* be at least $3^2 \times 5^2 = \mathbf{225}$.

S05SF6. Danny must travel north 4 avenues and east 3 streets. Thus he must travel 7 blocks in total. $\frac{7!}{4!3!} = \mathbf{35}$. (*nnnneee*)

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SOPH FROSH DIVISION **CONTEST NUMBER TWO** **SPRING 2005**

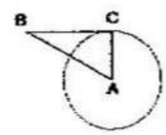
SOLUTIONS

S05SF7. $(x-2)(x^2+ax+b)=(x-2)(x-1)(x-c) \Rightarrow a=-(c+1),\ b=c$, so $a+b=-1$.

S05SF8. Call the rectangles' whose side runs parallel to the row of length 5, the width, and the rectangles' whose side runs parallel to the column of length 3, the length. The rectangles can have width 1, 2, 3, 4, 5 and length 1, 2, 3. There are 5 ways to have width of 1, 4 ways to have width of 2, 3 ways to have width of 3, 2 ways to have width of 4 and 1 way to have width of 5. Similarly, there are 3 ways to have length 1, 2 ways to have length 2, and 1 way to have length 3. Therefore we have 15 possible positions for widths, and 6 possible positions for lengths, generating **90** possible rectangles.

S05SF9. $\dfrac{a^{-1}+b^{-1}}{a+b}=\dfrac{\frac{1}{a}+\frac{1}{b}}{a+b}=\dfrac{\left(\frac{b+a}{ab}\right)}{a+b}=\dfrac{1}{ab}=\dfrac{1}{0.1}=\mathbf{10}$.

S05SF10. $\overline{BC} \perp \overline{AC}$. The area of the circle is 9π, and the sector is $\dfrac{\frac{3\pi}{2}}{9\pi}=\dfrac{1}{6}$ of the circle, so that angle *CAB* is 60 degrees. Therefore this is a 30-60-90 triangle and side *BC* must have length $3\sqrt{3}$. Therefore, the area is $\dfrac{9\sqrt{3}}{2}$.



S05SF11. If Bernadette had *n* children, then $\dfrac{3n}{4}+\dfrac{3}{4}=n \rightarrow 3n+3=4n \rightarrow n=\mathbf{3}$.

S05SF12. Let *S* be the sum of the digits of *N*. *S* is usually much less than *N*. In fact, if *N* has n digits, $N \geq 10^{n-1}$, and $S \leq 9n$. Hence $11 \cdot 9n \geq 11 \cdot S = N \geq 10^{n-1}$, or $10^{n-1} \leq 99n$. A quick check shows that this is true only for $n = 1,2,3$. Certainly *N* must have more than one digit, so it has either 2 or 3 digits.
Then let $N = 100a + 10b + c$, where *a, b, c* can be ay digit (including 0). Then $100a + 10b + c = 11a + 11b + 11c$, or $89a = b + 10c$. Then, since $a, b, c < 10$, we have $89a < 9 + 10 \cdot 9 = 99$, and a = 0 or 1. If $a = 0$, $b = c = 0$, which is not a natural number. If $a = 1$, $89 = 10c + b$, so $c = 8$ and $b = 9$. Thus the answer is **198.**

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SOPH FROSH DIVISION **CONTEST NUMBER THREE** **SPRING 2005**

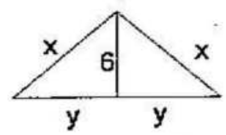
SOLUTIONS

S05SF13. In the time Luis rode the first half of the trip, he would have walked only 1/20 of the way. In the time he rode the second half, he would have walked ¼ of the way. Hence he would have walked 6/20 of the way, and he traveled 20/6 = **10/3** times as fast.

S05SF14. The altitude to the base of an isosceles triangle also bisects the side it intersects. Therefore:

$$2x+2y=36 \Rightarrow x+y=18, y^2+6^2=x^2 \Rightarrow x^2-y^2=36 \Rightarrow (x-y)(x+y)=36 \Rightarrow x-y=2$$

Therefore, we know that y is 8. From this, we can calculate the area as **48**.



S05SF15. If the average of 3 numbers is x, the sum of the numbers is 3x. If one of the numbers is y, the sum of the other two must be $3x - y$.

S05SF16. Trial and error is facilitated if we note that neither 2 nor 3 can be one of the given numbers. If 2 were given, the sum of 2 and two odd primes would be even. If 3 were given, either the other three primes have the same remainder upon division by 3, or two of them have remainders of 1 and 2 respectively. In either case, the sum of three of the numbers will be a multiple of 3. Hence the smallest number given must be at least 5. Trial and error shows that (5,7,17,19) has the desired property, and the sum of these four numbers is **48**. Any other set has a larger total sum.

S05SF17. The new perimeter is 11 times bigger, so each side is 11 times bigger. The area is 121 times bigger, so the percent increase is **12000%**.

S05SF18. Suppose Al's address is $100a + 10b + c$, where a, b, and c are digits. Then $a > b > c$, since Al's address cannot be rearranged to get a larger number. Bob's address must be $100a + 10c + b$, and adding, we find $200a + 11(b + c) = 1233$. Since $b + c < 20$, $11(b + c) < 220$. Subtracting this from the last equation, we find that $200a > 1013$, so $a > 5$ and $200a < 1233$, so $a < 7$. Thus $a = 6$, $11(b + c) = 33$, so $b + c = 3$. The only solution that fits the requirements of the problem is $b = 2$, $c = 1$.