New York City
Interscholastic
Mathematics
League

SENIOR A DIVISION **CONTEST NUMBER ONE** **SPRING 2005**
PART I: 10 minutes **NYCIML Contest One** **Spring 2005**

S05A1. The sum of the squares of the digits of the year 2005 is 29. Compute the next year whose sum of the squares of the digits is 29.

S05A2. Compute the number of distinct three-letter "words" that can be made from the letters in the word MATHEMATICS.

PART II: 10 minutes **NYCIML Contest One** **Spring 2005**

S05A3. In Toontown 2/3 of the men are married (to a woman), and 3/5 of the women are married (to a man) (one spouse per person). If p/q is the fractional part of the town that is married, compute p/q.

S05A4. A lottery consists of 15 different numbers. Jen chooses 5 of these numbers and then the 5 winning numbers are drawn at random. Compute the probability that none of Jen's numbers are picked as winning.

PART III: 10 minutes **NYCIML Contest One** **Spring 2005**

S05A5. Compute all real x: $\sqrt{3+\sqrt{x}}+\sqrt{4-\sqrt{x}}=\sqrt{7}$.

S05A6. Point E is chosen inside unit square $ABCD$ so that ABE is an equilateral triangle. Segment $\overline{BE}$ intersects diagonal $\overline{AC}$ at F, and segment $\overline{AE}$ intersects diagonal $\overline{BD}$ at G. Compute FG.

ANSWERS:
S05A1. **2034**
S05A2. **399**
S05A3. **12/19**
S05A4. **12/143**
S05A5. **16**
S05A6. $\mathbf{2-\sqrt{3}}$

New York City Interscholastic Mathematics League

SENIOR A DIVISION **CONTEST NUMBER TWO** **SPRING 2005**

PART I: 10 minutes **NYCIML Contest Two** **Spring 2005**

S05A7. $0^\circ \le x \le 180^\circ$. Compute all x if: $(\csc x + \cot x)(1 - \cos x) = \cos 23^\circ$.

S05A8. In ΔABC, median $AM = 4$, and median $BN = 3$. If $\overline{AM} \perp \overline{BN}$, compute the area of ΔABC.

PART II: 10 minutes **NYCIML Contest Two** **Spring 2005**

S05A9. Compute the minimum n such that $1+2+3+4+\ldots+n \ge 2{,}000{,}000$

S05A10. Diagonal AC divides trapezoid $ABCD$ into two isosceles triangles, with $AB = BC$ and $AC = CD$. If the bases of the trapezoid have lengths $BC = 8$ and $AD = 18$, compute the area of the trapezoid.

PART III: 10 minutes **NYCIML Contest Two** **Spring 2005**

S05A11. Define the *center* of the face of a cube as the intersection of the diagonals of that face. If every pair of centers of the faces of the cube are connected by line segments, compute the number of line segments there will be.

S05A12. Compute all integers n for which $\frac{4n-6}{3n+5}$ is an integer.

ANSWERS:

S05A7. **$67^\circ, 113^\circ$**

S05A8. **8**

S05A9. **2000**

S05A10. **$39\sqrt{7}$**

S05A11. **15**

S05A12. **-2, -1, -8, and 11**

New York City Interscholastic Mathematics League

SENIOR A DIVISION **CONTEST NUMBER THREE** **SPRING 2005**

PART I: 10 minutes **NYCIML Contest Three** **Spring 2005**

S05A13. $a = \sin x, b = \cos x, c = \tan x$. If $x = 1$ radian, list a, b, c, in descending order.

S05A14. All students at Clinton High School arrive by bus or train. Every 16 minutes, a train arrives, and a group of students get off and enter the school. Every 21 minutes, a bus arrives, and a group of students get off and enter the school. The principal watching the entrance counts the trains and busses by noting how many groups of students enter, but she cannot distinguish between two groups of students if the train and the bus arrive 6 minutes or less apart. Compute the number of groups of students the principal sees during a three hour period, starting with the arrival of a train and a bus simultaneously.

PART II: 10 minutes **NYCIML Contest Three** **Spring 2005**

S05A15. If one root of $2x^2 - 3bx + 1 = 0$ is $2 - \sqrt{3}$, compute the value of b.

S05A16. In ΔABC, $AB < BC$ and N is the midpoint of $\overline{AC}$. A line through N, parallel to $\overline{BC}$, intersects $\overline{AB}$ at M. The bisector of $\angle ABC$ intersects $\overline{MN}$ at E. Compute $m\angle AEB$, in degrees.

PART III: 10 minutes **NYCIML Contest Three** **Spring 2005**

S05A17. Compute the coefficient of $x^2y^2z^2w^2u^2$ in the expanded expression. $(x+y+z+w+u)^{10}$.

S05A18. $\sqrt{3+\sqrt{13+\sqrt{48}}} = a + \sqrt{b}$, compute $\frac{a+b}{ab}$.

ANSWERS:

S05A13. $c > a > b$ or c, a, b

S05A14. **14**

S05A15. $\frac{6-\sqrt{3}}{3}$ or $2-\frac{\sqrt{3}}{3}$

S05A16. **90**

S05A17. **113400**

S05A18. **4/3**

SENIOR A DIVISION **CONTEST NUMBER FOUR** **SPRING 2005**

PART I: 10 minutes **NYCIML Contest Four** **Spring 2005**

S05A19. Compute all real x: $x^{\frac{1}{2}(\log_5 x-1)} = 5$.

S05A20. Inside a box are 10 blue marbles and 5 red marbles. There are also an infinite number of extra blue marbles, not in the box. Two marbles from the box are chosen at random and removed. If the two are of the same color, they both are removed, and one new blue marble is put in the box. If they are of opposite colors, only the blue marble is removed. This process continues until one last marble is left inside the box. Compute the probability that this last marble is red.

PART II: 10 minutes **NYCIML Contest Four** **Spring 2005**

S05A21. Two adjacent sides of a unit square are used as diameters to construct circles O and P. Compute the area of the region common to the interiors of both circles.

S05A22. We define the new operation "*" by $a*b = a+b-ab$. If $a*b=25$, and a and b are integers, compute the greatest possible value of $a+b$.

PART III: 10 minutes **NYCIML Contest Four** **Spring 2005**

S05A23. Compute the number of two-digit positive integers that are divisible by the product of their digits.

S05A24. Consider the seventh degree polynomial equation: $x^7-28x^6+x^4+ax^3+bx^2+c=0$, whose roots are all real and form an arithmetic progression, and some of the roots are irrational. Compute the greatest root of the equation.

ANSWERS:

S05A19. **25, 1/5**
S05A20. **1**
S05A21. $\frac{\pi-2}{8}$
S05A22. **25**
S05A23. **5**
S05A24. $4+6\sqrt{6}$

SENIOR A DIVISION | **CONTEST NUMBER FIVE** | **SPRING 2005**

PART I: 10 minutes | **NYCIML Contest Five** | **Spring 2005**

S05A25. For integer x and prime number P, $x^3-27=P$. Compute all P.

S05A26. If $(a+bi)^6=3+4i$, compute $(b-ai)^6$.

PART II: 10 minutes | **NYCIML Contest Five** | **Spring 2005**

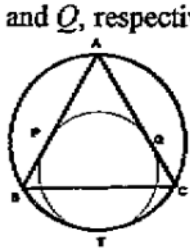
S05A27. The average of n consecutive odd integers is n. If the largest of the integers is 2005, compute n.

S05A28. Three numbers form a geometric progression with sum 14, and the sum of their squares is 364. Compute the three numbers.

PART III: 10 minutes | **NYCIML Contest Five** | **Spring 2005**

S05A29. $\cos^2 x=\frac{\sqrt{2}}{2}$, $\csc^2 x=a+b\sqrt{2}$, where a and b are integers. Compute (a,b).

S05A30. Equilateral triangle ABC is inscribed in a circle. A second circle is tangent internally to the circumcircle at T and tangent to $\overline{AB}$ and $\overline{AC}$ at points P and Q, respectively. If $BC=15$, compute PQ.



ANSWERS:

S05A25. **37**

S05A26. **$-3-4i$**

S05A27. **1003**

S05A28. **2, -6, 18**

S05A29. **(2,1)**

S05A30. **10**

New York City Interscholastic Mathematics League

SENIOR A DIVISION **CONTEST NUMBER ONE SOLUTIONS** **SPRING 2005**

S05A1. Listing the groups of four digits that have the property we find only $\{5,2,0,0\}$ and $\{4,3,2,0\}$. The next year will be **2034**.

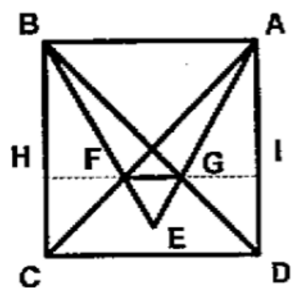
S05A2. There are 8 different letters, 3 of which are repeated twice. There are $8\times7\times6=336$ words with different letters, and $3\times3\times7=63$ words with two letters the same, for a total of **399** different words.

S05A3. Let M be the number of married couples. Then there are $\dfrac{M}{2}$ unmarried men, and $\dfrac{2M}{3}$ unmarried women, for a total of $2M+\dfrac{M}{2}+\dfrac{2M}{3}=\dfrac{19M}{6}$ people living in town. The fraction that is married is $2M/\left(\dfrac{19M}{6}\right)=\dfrac{12}{19}$

S05A4. There are $\dbinom{15}{5}$ ways of picking the winning numbers. Of these, $\dbinom{10}{5}$ do not include Jen's numbers. $\dbinom{10}{5}/\dbinom{15}{5}=\dfrac{12}{143}$

S05A5 Let $a=3+\sqrt{x}$ and $b=4-\sqrt{x}$ Then $a+b=7$, and we have $\sqrt{a}+\sqrt{b}=\sqrt{a+b}$. Squaring both sides gives $a+b+2\sqrt{ab}=a+b$ Thus, either a or b must be 0. If $3+\sqrt{x}=0$, there is no real solution. If $4-\sqrt{x}=0$, x must be **16**.

S05A6. Extend $\overline{FG}$ to meet $\overline{BC}$ and $\overline{AD}$ at H and I, respectively. Let $FG=x$. Then $FH=GI=\dfrac{1-x}{2}$. ΔCFH is an isosceles right triangle, so $CH=FH=\dfrac{1-x}{2}$. ΔBFH is a 30-60-90 triangle, so $BH=FH\sqrt{3}=\dfrac{\sqrt{3}(1-x)}{2}$. But $BH+CH=1$, and solving $\dfrac{1-x}{2}+\dfrac{\sqrt{3}(1-x)}{2}=1$ for x we obtain $x=FG=\mathbf{2-\sqrt{3}}$



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SENIOR A DIVISION **CONTEST NUMBER TWO SOLUTIONS** **SPRING 2005**

S05A7.

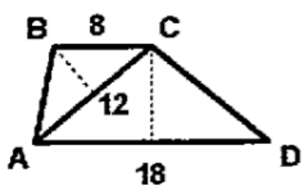
$$(\csc x+\cot x)(1-\cos x)=\frac{1+\cos x}{\sin x}(1-\cos x)=\frac{1-(\cos x)^2}{\sin x}=\frac{(\sin x)^2}{\sin x}=\sin x=\cos 23^\circ.$$

Thus, $x=67^\circ, 113^\circ$.

S05A8. Let X = point of intersection of $\overline{AM}$ and $\overline{BN}$. The medians of a triangle divide each other in the ratio 2:1. Thus $AX = 8/3$, $BX = 2$, and the area of triangle $ABX = 8/3$. We can now relate the area of triangle ABX to that of triangle ABC. We use absolute value to denote area. Since triangles ABX, BXM have equal altitudes from B, $|ABM|=(3/2)|ABX|$. Since triangles ABM, AMC have equal altitudes from A, their areas are equal. Hence $|ABC|=3|ABX|$, or 8.

S05A9. $\frac{n(n+1)}{2}\geq 2{,}000{,}000$, so $n(n+1)\geq 4{,}000{,}000=2000^2$, so $n\geq$ **2000**

S05A10. $\overline{AC}$ acts like a transversal, so $\angle BCA\cong\angle CAD$. Since their base angles are congruent, ΔABC and ΔACD are similar and $\frac{BC}{AC}=\frac{AC}{AD}$, so $AC=12$. The height from C to $\overline{AD}$ is $\sqrt{12^2-9^2}=3\sqrt{7}$, so the area of ΔACD is $\frac{18(3\sqrt{7})}{2}=27\sqrt{7}$. Similarly, the area of ΔABC is $12\sqrt{7}$ and the area of the trapezoid is $27\sqrt{7}+12\sqrt{7}=39\sqrt{7}$.



S05A11. A cube has six faces, so there are $\binom{6}{2}=15$ such segments. These segments form a regular octahedron, together with its diagonals.

S05A12. If $A=\frac{4n-6}{3n+5}$ is an integer, then so is $3A=\frac{12n-18}{3n+5}=4+\frac{-38}{3n+5}$. Thus $3n+5$ is a factor of -38. Setting $3n+5$ equal to ±1, ±2, ±19, and ±38, we find that the only integer solutions for n are -2, -1, -8, and 11. Substituting each into $A=\frac{4n-6}{3n+5}$ we find that all produce integers. Answer: **-2, -1, -8, and 11.**

New York City Interscholastic Mathematics League

SENIOR A DIVISION **CONTEST NUMBER THREE** **SPRING 2005**

SOLUTIONS

S05A13. If $x = \pi/4 < 1$, then $\sin x = \cos x$. Between $\pi/4$ and $\pi/2$, $\sin x$ increases while $\cos x$ decreases. Hence $\sin 1 > \cos 1$. Then $\tan 1 = \sin 1/\cos 1 > 1$, so $\tan 1 > \sin 1$. Thus $\boldsymbol{c > a > b}$.

S05A14. Starting at $t = 0$, when the first train and bus arrive, we can list the arrival times:

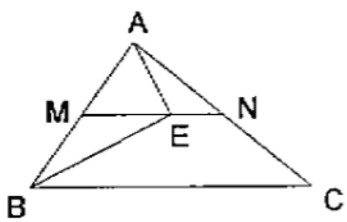
Train: 0, 16, 32, 48, 64, 80, 96, 112, 128, 144, 160, 176

Bus: 0, 21, 42, 63, 84, 105, 126, 147, 168

There are 12 trains and 9 busses, but we must subtract those trains and busses that arrive 6 minutes or less apart. There are seven of these, and $12 + 9 - 7 =$ **14** groups of students distinguishable to the principal.

S05A15. Since $2-\sqrt{3}$ is a root, $2\left(2-\sqrt{3}\right)^2 - 3b\left(2-\sqrt{3}\right) + 1 = 0$. Solving for b,

$b = \mathbf{\dfrac{6-\sqrt{3}}{3}}$ or $\mathbf{2 - \dfrac{\sqrt{3}}{3}}$.



S05A16. M is the midpoint of $\overline{AB}$. $m\angle EBC = m\angle MEB = m\angle MBE$ Thus ΔMBE is isosceles, with $MB = ME$, but $MB = MA$, therefore $m\angle AEB =$ **90**.

S05A17. The coefficient of $(x+1)^n$ exists in the n^{th} row of the Pascal triangle. A similar combinatorial identity exists for any expansion, the coefficient of $x^2y^2z^2w^2u^2$ in the expansion of $(x+y+z+w+u)^{10}$ is

$\binom{10}{2}\cdot\binom{8}{2}\cdot\binom{6}{2}\cdot\binom{4}{2}\cdot 1$ (akin to picking out of 10 variables 2 x's, and then picking 2 y's from the remaining 8 and so on.) Therefore the answer is **113400**.

S05A18.

$$\sqrt{3+\sqrt{13+\sqrt{48}}} = \sqrt{3+\sqrt{13+4\sqrt{3}}} = \sqrt{3+\sqrt{\left(1+2\sqrt{3}\right)^2}} = \sqrt{3+\left(1+2\sqrt{3}\right)} = \sqrt{4+2\sqrt{3}} = 1+\sqrt{3}$$

Therefore $\dfrac{a+b}{ab} =$ **4/3.**

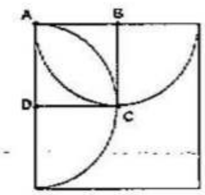
S05A19. $\log_5 x^{\frac{1}{2}(\log_5 x-1)} = \log_5 5 = 1.$

$$\frac{1}{2}(\log_5 x - 1)\log_5 x = 1$$

Let $A = \log_5 x \rightarrow \frac{1}{2}(A-1)A = 1 \rightarrow A = 2$ or $A = -1 \rightarrow x = 25, \frac{1}{5}$.

S05A20 Each time two different color marbles is chosen, the number of marbles inside the box is reduced by one. No matter how this happens, the number of red marbles inside the box remains odd. Hence the last marble must be red (since 1 is odd), and the required probability is 1.

S05A21. The area of the common region is the area of sector ACD plus the area of sector ABC minus the area of square $ABCD$.



$$2\left(\frac{\pi}{16}\right) - \frac{1}{4} = \frac{\pi - 2}{8}.$$

S05A22. Since $(a-1)(b-1) = ab - a - b + 1$, $a*b - 1 = -(a-1)(b-1)$, which relates the new operation to ordinary multiplication. Here, we have $-(a-1)(b-1) = 24$, and we can construct a table of "* factors" of 25, based on the usual factors of 24:

Factors of 24		a	b
1	24	2	-23
		0	25
2	12	3	-11
		-1	13
3	8	4	-7
		-2	9
4	6	5	-5
		-3	7

Each pair of factors of 24 will generate two pairs of "* factors" of 25, and by the symmetry of the "*" operation, we need not consider the other factorizations of 24. The table shows the maximal value of a + b to be 25.

S05A23. Let the ten's digit be t and the unit's digit be u. $10t + u$ is divisible by $t \cdot u$. Thus, t divides u and u divides $10t$. Let $u = kt$. Since kt divides $10t$, k divides 10 and $k = 1, 2, 5$. For $k = 1$, $10t + u = 11$, for one solution, for $k = 2$, we have 12, 24, 36, for three solutions, and for $k = 5$, we have 15 for 5 solutions in total.

S05A24. The sum of the roots of the equation is 28, and since there are seven distinct roots, the average root must be 4. If d is the common difference of the arithmetic progression they form, the roots can then be written as:

$$4-3d,\ 4-2d,\ 4-d,\ 4,\ 4+d,\ 4+2d,\ 4+3d.$$

The sum of the roots "taken two at a time" is zero. We can use this to find the sum of the squares of the roots. Since the expressions for the roots are written symmetrically around the middle root, this will allow us to solve for d.

$$(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7)^2 =$$
$$r_1^2 + r_2^2 + r_3^2 + r_4^2 + r_5^2 + r_6^2 + r_7^2 + 2(r_1r_2 + r_1r_3 + r_1r_4 + \ldots + r_6r_7)$$

or $\left(\sum_{i=1}^{7} r_i\right)^2 = \sum_{i=1}^{7} r_i^2 + 2\sum_{1 \le i \le j \le 7} r_i r_j$, which expresses the sum of the squares of the roots in terms of the coefficients of the equation. Substituting, we find

$$\left(\sum_{i=1}^{7} r_i\right)^2 = 2(16 + 9d^2) + 2(16 + 4d^2) + 2(16 + d^2) + 16 = 112 + 28d^2$$

$28^2 = 112 + 28d^2$, and $d^2 = 168/7 = 24$, so $d = \sqrt{24} = 2\sqrt{6}$. This makes the largest root $4 + 3d = 4 + 6\sqrt{6}$.

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SENIOR A DIVISION **CONTEST NUMBER FIVE** **SPRING 2005**

SOLUTIONS

S05A25. $P = x^3 - 3^3 = (x-3)(x^2+3x+9) \rightarrow x-3 = 1$ or P. If $x = P+3 \rightarrow (x^2+3x+9) = 1$, so P cannot be prime. If $x = 4 \rightarrow P = 4^3 - 27 = \mathbf{37}$.

S05A26. $(b-ai)^6 = ((a+bi)(-i))^6 = (a+bi)^6(-i)^6 = (3+4i)(-1) = \mathbf{-3-4i}$.

S05A27. Let a be the smallest of the integers. The average of the terms of an arithmetic sequence is the average of the largest and the smallest terms, so $n = \frac{a+2005}{2}$. The number of terms will be $n = \frac{2005-a}{2}+1$. Setting these equal we have $a = 1$ and $n = \mathbf{1003}$.

S05A28. If the middle number is a, and the common ratio r, then the numbers can be represented by a/r, a, and ar. Then:

$(i) a/r + a + ar = 14$

$(ii) a^2/r^2 + a^2 + a^2r^2 = 364$

Squaring (i), we find $a^2/r^2 + 3a^2 + a^2r^2 + 2a^2/r + 2a^2r = 196$ and subtracting (ii), we find $a^2/r + a^2 + a^2r = -84$. Since $a(a/r + a + ar) = a^2/r + a^2 + a^2r$, we can divide to get $a = -84/14 = -6$. Then $-6/r + -6 + -6r = 14$, or $-6(r + 1/r) = 20$, and $r + 1/r = -10/3$. It quickly follows that $r = -3$ or $-1/3$. Both values give the same progression, but in opposite order, so the numbers are **2, -6, 18**.

S05A29. $\sin^2 x = 1 - \cos^2 x = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$. So $\csc^2 x = \frac{2}{2-\sqrt{2}} = \frac{4+2\sqrt{2}}{2} = 2+\sqrt{2}$. Thus the answer is $\mathbf{(2,1)}$.

S05A30. (see diagram) Let O and D be the centers of the small and large circles, respectively. Note that ΔOQA is a 30-60-90 triangle, so $OA = 2OQ = 2OT$, so $AT = 3OT$, the diameter of circle D is three times the radius of circle O, so the radii of the circles are in a ratio of 3:2. Furthermore, note that because $m\angle AOQ = 60°$, $m\angle TOQ = 120°$, $m\angle OTQ = 30°$, and $m\angle PTQ = 60°$, so ΔPTQ is an equilateral triangle inscribed in the small circle. Since the ratio of the radii of the circles is 3:2, that will also be the ratio of the sides of their inscribed triangles. $BC : PQ = 3:2$, and since $BC = 15$, $PQ = \mathbf{10}$.