New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER ONE** **FALL 2004**

PART I: 10 minutes **NYCIML Contest One** **Fall 2004**

F04B01. Express $\sin 345^\circ$ in simplest radical form.

F04B02. If $\log_{10} x = 2 - 2\log_{10} 2$, compute x.

PART II: 10 minutes **NYCIML Contest One** **Fall 2004**

F04B03. Compute: $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{99}\right)\left(1+\frac{1}{100}\right)$.

F04B04. Let $N = 3^a + 7^b$. If a and b are integers, not necessarily different, chosen at random from 1 to 100 inclusive, compute the probability that the units digit of N is 4.

PART III: 10 minutes **NYCIML Contest One** **Fall 2004**

F04B05. Express in terms of N the average of the first N positive integers.

F04B06. An isosceles triangle has sides 10, 12, and 12. Compute the length of the radius of the circumscribed circle.

ANSWERS

1. $\frac{\sqrt{2}-\sqrt{6}}{4}$
2. 25
3. $\frac{101}{2}$ or 50.5 or $50\frac{1}{2}$
4. $\frac{3}{16}$
5. $\frac{N+1}{2}$
6. $\frac{72\sqrt{119}}{119}$

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SENIOR B DIVISION — CONTEST NUMBER TWO — FALL 2004

PART I: 10 minutes — **NYCIML Contest Two** — **Fall 2004**

F04B07. If the sum of the first 100 positive integers is subtracted from the sum of the first 100 positive odd integers, compute the result.

F04B08. Compute the three digit number whose middle digit is 0, such that when the number is divided by 11, the result is the sum of the squares of the original number's digits.

PART II: 10 minutes — **NYCIML Contest Two** — **Fall 2004**

F04B09. There are 100 students in the sophomore class of Bawne High School. 43 students take biology, 35 take physics, 42 take chemistry, 12 take physics and chemistry, 15 take biology and physics, 17 take biology and chemistry and 7 take all 3 sciences. How many students take neither biology, chemistry, or physics?

F04B10. Compute $\sqrt{5+\sqrt{5+\sqrt{5+\cdots}}}$

PART III: 10 minutes — **NYCIML Contest Two** — **Fall 2004**

F04B11. Compute the positive value of $\tan x$ if $25\tan x = 24\sec x$.

F04B12. The sum and the product of the roots of a quadratic equation, not necessarily different, are chosen from the set $\{1,2,3,\cdots,9,10\}$. Compute the probability that the roots are real.

ANSWERS

7.	4950	10.	$\frac{1+\sqrt{21}}{2}$
8.	803	11.	$\frac{24}{7}$
9.	17	12.	$\frac{31}{50}$

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SENIOR B DIVISION **CONTEST NUMBER THREE** **FALL 2004**

PART I: 10 minutes **NYCIML Contest Three** **Fall 2004**

F04B13. The average of 12 numbers is 12. The average of 8 of these numbers is 8. Compute the average of the other 4 numbers.

F04B14. Compute the number of positive integral solutions (x, y) for $3x+4y=1000$.

PART II: 10 minutes **NYCIML Contest Three** **Fall 2004**

F04B15. A triangle has vertices at $(9,0),(0,6)$, and $(0,0)$. Compute the coordinates of the point of intersection of the three medians of the triangle.

F04B16. In a circle, $\overline{AB}$ and $\overline{AC}$ are chords, $AB=2$, $AC=4$, and $m\angle BAC=30^\circ$. A perpendicular from B to $\overline{AC}$ is extended until it intersects the circle at D. Compute BD.

PART III: 10 minutes **NYCIML Contest Three** **Fall 2004**

F04B17. Compute the sum of all positive 4-digit integers whose digits are different prime numbers.

F04B18. Compute the number of positive integers x, $3<x<50$, such that 3^x-x^3 is a multiple of 6.

ANSWERS

13. 20
14. 83
15. (3, 2)
16. $4\sqrt{3}-2$
17. 113322
18. 7

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SENIOR B DIVISION **CONTEST NUMBER FOUR** **FALL 2004**

PART I: 10 minutes **NYCIML Contest Four** **Fall 2004**

F04B19. An equilateral triangle has vertices $(-4,0)$ and $(4,0)$. Compute the coordinates of all possible third vertices.

F04B20. The sides of an isosceles triangle are 4, 4, and 5. Compute the length of the altitude to one of the legs.

PART II: 10 minutes **NYCIML Contest Four** **Fall 2004**

F04B21. Compute the sum of the coefficients of the polynomial $(2x+3y)^4$ when it is expanded.

F04B22. Compute the number of base b's, $1<b<100$, such that 1100_b is a perfect square.

PART III: 10 minutes **NYCIML Contest Four** **Fall 2004**

F04B23. Compute the value of $(1+i)^{20}-(1-i)^{20}$.

F04B24. Compute the length of a side of a regular octagon which is inscribed in a circle with radius 1.

ANSWERS

19.	$\left(0,4\sqrt{3}\right),\left(0,-4\sqrt{3}\right)$	22.	9
20.	$\frac{5\sqrt{39}}{8}$	23.	0
21.	625	24.	$\sqrt{2-\sqrt{2}}$

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **FALL 2004**

PART I: 10 minutes **NYCIML Contest Five** **Fall 2004**

F04B25. There are 10 black and 8 white socks in a drawer. If 2 socks are chosen at random, compute the probability that they are the same color.

F04B26. Compute all x which satisfy: $\sqrt{x+\frac{2}{3}}=x\sqrt{\frac{2}{3}}$.

PART II: 10 minutes **NYCIML Contest Five** **Fall 2004**

F04B27. Compute the number of positive integral divisors of 12^6.

F04B28. Compute all values of x, $\left(0^\circ \le x \le 360^\circ\right)$ such that $\sin 2x < \sin x$.

PART III: 10 minutes **NYCIML Contest Five** **Fall 2004**

F04B29. A pizza can have toppings of mushrooms, peppers, ham, anchovies, and pineapples. Maria puts at least one topping on her pizza, or as many as all five. How many different varieties can she make?

F04B30. Moe, Larry, and Curly, in that order, take turns rolling a pair of dice. The first boy who rolls a total of 9 on his two dice is the winner. The game continues until one boy wins. Compute the probability that Larry wins.

ANSWERS

25.	$\frac{73}{153}$	28.	$60° < x < 180°, 300° < x < 360°$
26.	2	29.	31
27.	91	30.	$\frac{72}{217}$

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SENIOR B DIVISION **CONTEST NUMBER ONE** **Fall 2004**

SOLUTIONS

F04B1

$$\sin 345^\circ = \sin(300^\circ + 45^\circ) = \sin 300^\circ \cos 45^\circ + \cos 300^\circ \sin 45^\circ = -\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2} = \mathbf{\frac{\sqrt{2}-\sqrt{6}}{4}}$$

F04B2 $\log_{10} x = \log_{10} 100 - \log_{10} 4 = \log_{10} \frac{100}{4} = \log_{10} 25 \therefore x = \mathbf{25}$

F04B3 $\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\frac{6}{5}\cdots\frac{100}{99}\frac{101}{100}$. This telescopes to $\mathbf{\frac{101}{2}}$.

F04B4 The powers of 3 have a units digit of 3, 9, 7, and 1. The powers of 7 have a units digit of 7, 9, 3, and 1. There are an equal number of each, and 4×4 or 16 ways to combine the possible results. The only ways to produce a 4 as the units digit are $1+3, 3+1$, or $7+7$. Thus the probability is $\mathbf{\frac{3}{16}}$.

F04B5 The sum of the first N integers is $\frac{N(N+1)}{2}$. The average is

$$\frac{\frac{N(N+1)}{2}}{N} = \mathbf{\frac{N+1}{2}}$$

F04B6 The height $h = \sqrt{119}$

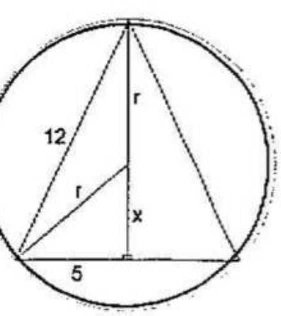
$$r^2 = 25 + x^2 = 25 + (h-r)^2 = 25 + h^2 - 2hr + r^2 = r^2$$

$$25 + h^2 - 2hr = 0$$

$$25 + 119 - 2r\sqrt{119} = 0$$

$$2r\sqrt{119} = 144$$

$$r = \frac{72}{\sqrt{119}} = \mathbf{\frac{72\sqrt{119}}{119}}$$



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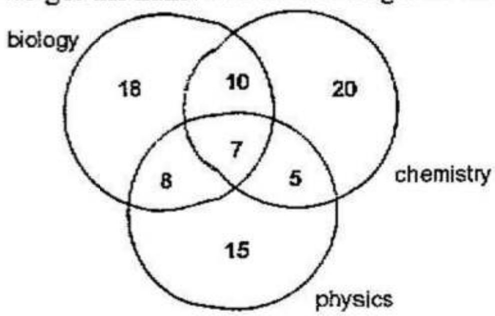
SENIOR B DIVISION **CONTEST NUMBER TWO** **Fall 2004**

SOLUTIONS

F04B7 The sum of the first 100 positive integers is $\frac{100(101)}{2}=5050$. The sum of the first 100 odd integers is $(100)^2=10000$. $10000-5050=\mathbf{4950}$.

F04B8 Since the number is divisible by 11, the sum of the first digit and last digit is 11. the only possibilities are 209, 308, 407, 506, 605, 704, 803, and 902. The quotients when divided by 11 are 19, 28, 37, 46, 55, 64, 73, and 82. $8^2+3^2=73$ gives **803** as the only solution.

F04B9 A Venn diagram gives the easiest solution. Working from the inside out, we get the numbers on the diagram. That leaves **17** people who take no science.



F04B10 Let $x=\sqrt{5+\sqrt{5+\sqrt{5+\cdots}}}$. $x=\sqrt{5+x}$. $x^2=5+x$. $x^2-x-5=0$. $x=\frac{1\pm\sqrt{21}}{2}$. Rejecting the negative root, $x=\mathbf{\frac{1+\sqrt{21}}{2}}$.

F04B11 $25\frac{\sin x}{\cos x}=24\frac{1}{\cos x}$. $\sin x=\frac{24}{25}$. By Pythagorean theorem, $\tan x=\mathbf{\frac{24}{7}}$.

F04B12 Let the equation be: $x^2-ax+b=0$. For the roots to be real, $a^2-4b\geq 0$ so $b\leq\frac{a^2}{4}$. Start with $a=10,9,\ldots$ the number of b 's that will work for each a is 10,10,10,10,9,6,4,2,1,0. 62 pairs work out of a total of 100 possibilities. $\frac{62}{100}=\mathbf{\frac{31}{50}}$.

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SENIOR B DIVISION **CONTEST NUMBER THREE** **Fall 2004**

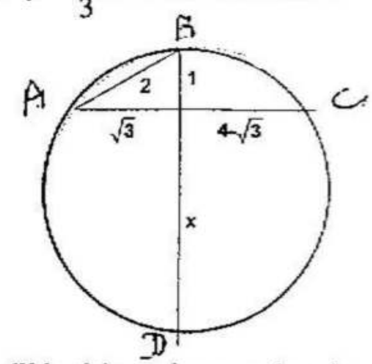
SOLUTIONS

F04B13 The sum of those 8 is 64. The sum of the 12 numbers is 144. The sum of the other 4 is 80. $\frac{80}{4} = \mathbf{20}$.

F04B14 $y = \frac{1000 - 3x}{4} = 250 - \frac{3x}{4}$. x must be a multiple of 4, from 4 to 332. $4 = 4 \times 1$, $332 = 4 \times 83$. **83** numbers.

F04B15 The median with endpoints $(9,0)$ and $(0,3)$ has equation $y = -\frac{1}{3}x + 3$. The median with endpoints $(0,0)$ and $\left(\frac{9}{2}, 3\right)$ has equation $y = \frac{2}{3}x$. They intersect at $\mathbf{(3,2)}$.

F04B16 $1 \times x = \sqrt{3}\left(4 - \sqrt{3}\right)$. $x = 4\sqrt{3} - 3$. $BD = x + 1 = \mathbf{4\sqrt{3} - 2}$.



F04B17 The only primes are 2, 3, 5, and 7. There will be 24 numbers, and each digit will appear 6 times in each column. $(6)(17)(1000 + 100 + 10 + 1) = 113322$.

F04B18 3^x will always leave a remainder of 3 when divided by 6. x^3 will leave remainders of $1,2,3,4,5,0,1,2\ldots$ when x = 1, 2, 3, 4, 5, 6, 7, 8, ... Therefore, the only numbers which will leave a remainder of 0 when $3^x - x^3$ is divided by 6 are the multiples of 3 which are not multiples of 6. These values of x are 9, 15, 21, 27, 33, 39, and 45. There are 7 numbers.

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SENIOR B DIVISION **CONTEST NUMBER FOUR** **Fall 2004**
SOLUTIONS

F04B19 Since the side is 8, the altitude is $4\sqrt{3}$, either on the positive or negative y-axis. $(\mathbf{0}, 4\sqrt{3})$, $(\mathbf{0}, -4\sqrt{3})$.

F04B20 Using the Pythagorean theorem, the altitude to the base is $\frac{\sqrt{39}}{2}$. Since the product of a side and the altitude to it is a constant in any triangle, $5\frac{\sqrt{39}}{2} = 4h$.

$h = \frac{\mathbf{5\sqrt{39}}}{\mathbf{8}}$.

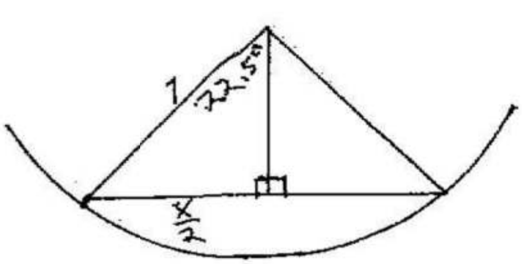
F04B21 The sum of the coefficients can be found by substituting 1 for each variable $(2 \cdot 1 + 3 \cdot 1)^4 = 5^4 = \mathbf{625}$.

F04B22 $1100_b = b^3 + b^2 = b^2(b+1)$. For that to be a perfect square, $b+1$ must be a perfect square. That will work for b=3,8,15,24,35,48,63,80,99. There are **9** numbers.

F04B23
$$(1+i)^{20} = (2i)^{10}$$
$$(1-i)^{20} = (-2i)^{10}$$
$$(2i)^{10} - (-2i)^{10} = \mathbf{0}$$

F04B24 Since the central angle $= 45^\circ$, $\sin 22.5^\circ = \frac{x}{2} = \sqrt{\frac{1-\cos 45^\circ}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

and $x = \sqrt{\mathbf{2} - \sqrt{\mathbf{2}}}$.



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SENIOR B DIVISION **CONTEST NUMBER FIVE** **Fall 2004**

SOLUTIONS

F04B25 Either they are both black, $\frac{10}{18}\cdot\frac{9}{17}$ or both white $\frac{8}{18}\cdot\frac{7}{17}$.
$\frac{90}{306}+\frac{56}{306}=\frac{146}{306}=\mathbf{\frac{73}{153}}$.

F04B26 Square both sides:
$x+\frac{2}{3}=\frac{2}{3}x^2 \rightarrow 2x^2-3x-2=0 \rightarrow x=-\frac{1}{2}, x=2$. $x=-\frac{1}{2}$ is rejected, so $x=\mathbf{2}$.

F04B27 The number of factors of $p_1^{e_1}p_2^{e_2}\ldots$ is $(e_1+1)(e_2+1)\ldots$ if p_k are the prime factors. $12^6=2^{12}3^6 \rightarrow 13\cdot 7=\mathbf{91}$.

F04B28 $\sin 2x-\sin x<0 \rightarrow 2\sin x\cos x-\sin x<0 \rightarrow \sin x(2\cos x-1)<0$. If
$\sin x<0$ and $2\cos x-1>0 \rightarrow \mathbf{300^\circ < x < 360^\circ}$
$\sin x>0$ and $2\cos x-1<0 \rightarrow \mathbf{60^\circ < x < 180^\circ}$

F04B29 The number of subsets of 5 objects is $2^5=32$. However, this includes the empty set which is excluded by the problem. 32-1=**31.**

F04B30 The probability of rolling a 9 is 4/36=1/9. For Larry to win on the first round, Moe must make a roll other than 9. If no one wins on the first round, Larry's probability of winning on the second round is $\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{1}{9}$. This will lead to an infinite geometric series. Using the formula, $S=\frac{a}{1-r}=\frac{\frac{8}{81}}{1-\frac{512}{729}}=\mathbf{\frac{72}{217}}$.