**F04A1.**     Compute the area of a regular 12-sided polygon that can be inscribed in a circle of radius 4.

**F04A2.**     Find all ordered pairs (x, y) of real numbers such that

$$3^{x^2-2xy} = 1 \text{ and } 2\log_3 x = \log_3(y+3)$$

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**PART II: 10 minutes**          **NYCIML Contest One**          **Fall 2004**

**F04A3.**     Compute the degree measure of the smaller angle between the two hands of a clock at 12:24.

**F04A4.**     Compute the real value of $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$

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**PART III: 10 minutes**          **NYCIML Contest One**          **Fall 2004**

**F04A5.**     Marina has $81 in $2 and $5 bills. If Marina has a total of $x$ bills, compute the number of possible different values for $x$.

**F04A6.**     A certain coin is weighted in such a way that the probability of getting a head on the $n$th flip is $\dfrac{1}{2^n}$. If the probability of getting 9 heads and 1 tail in 10 flips is $\dfrac{a}{2^{55}}$, compute $a$.

**ANSWERS:**
F04A1. **48**
F04A2. **(2,1)**
F04A3. **132 or 132°**
F04A4. **1**
F04A5. **8**
F04A6. **2036**

**F04A7.** The six faces of a cubic wooden block were painted blue, and then the block was cut into 125 small cubic blocks of identical size. Compute the number of small blocks with an odd number of faces painted blue.

**F04A8.** Compute all values of $x$ for which $|x^2 - x - 3| = 1 + x$.

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**PART II: 10 minutes**   **NYCIML Contest Two**   **Fall 2004**

**F04A9.** If $z$ is a complex number for which $z^2 = 2z - 2$, compute $z^4$.

**F04A10.** All the natural numbers are written in sequence 12345678910111213114... Compute the $2004^{\text{th}}$ digit of the sequence.

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**PART III: 10 minutes**   **NYCIML Contest Two**   **Fall 2004**

**F04A11.** Compute the greatest common factor of $(2003^2 - 1)$ and $(2005^2 - 9)$.

**F04A12.** In $\triangle ABC$, the bisector of $\angle B$ intersects $\overline{AC}$ at $D$. $E$ is the foot of the altitude from $A$ to $\overline{BC}$. Perpendiculars from $D$ and $E$ to $\overline{AB}$ meet $\overline{AB}$ at $F$ and $G$, respectively. If $AF = GB = 18$ and $FG = 14$, compute the area of $\triangle ABC$.

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**ANSWERS:**
F04A7. **62**
F04A8. $\sqrt{2}$, $1 + \sqrt{5}$
F04A9. **-4**
F04A10. **4**
F04A11. **8008**
F04A12. $\dfrac{2000}{3}$

**F04A13.**    Solve for all real values of $x$: $\log_8 x + \log_x 8 = \dfrac{10}{3}$

**F04A14.**    In the annual Elfland Lottery a natural number is picked at random, with the probability of picking the number $n$ given by $\dfrac{1}{2^n}$. Compute the probability that the number picked is even but not divisible by 5.

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**PART II: 10 minutes**        **NYCIML Contest Three**       **Fall 2004**

**F04A15.**    If $r$ and $s$ are the roots of $x^2 - 2x + 2 = 0$, compute $r^4 + s^4$.

**F04A16.**    In circle $O$ two perpendicular chords of lengths 16 and 20 are drawn. Let $A$ denote the intersection of the chords. If $AO = 9$, compute the radius of the circle.

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**PART III: 10 minutes**        **NYCIML Contest Three**       **Fall 2004**

**F04A17.**    If $\cos x° = \cos 40° + \cos 80°$ $\left(0° \le x \le 90°\right)$, compute $x$.

**F04A18.**    If, for all real numbers $x$ and $y$, the function $f$ satisfies
$$f(x+y) = \left(f(x)\right)^2 + \left(f(y)\right)^2,$$ compute all possible values of $f(2004)$.

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**ANSWERS:**
F04A13.  **2, 512**
F04A14.  $\dfrac{340}{1023}$
F04A15.  **-8**
F04A16.  $\dfrac{7\sqrt{10}}{2}$
F04A17.  **20**
F04A18.  **0,** $\dfrac{1}{2}$

**F04A19.** Compute the sum of all 3-digit positive integers that have only odd digits.

**F04A20.** Compute $\sqrt{3+2\sqrt{2}}+\sqrt{6-4\sqrt{2}}$ .

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**PART II: 10 minutes**        **NYCIML Contest Four**        **Fall 2004**

**F04A21.** A public pool has three pumps attached. Pump A can fill the pool in 8 hours. Pump B can fill the pool in 12 hours. Pump C can fill the pool in 15 hours. Compute the number of hours it would take to fill the pool with all three pumps operating simultaneously.

**F04A22.** Compute the minimum value of $\dfrac{x^2+3x+3}{(x+3)^2}$ for real $x$.

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**PART III: 10 minutes**        **NYCIML Contest Four**        **Fall 2004**

**F04A23.** Compute $k$ if three of the roots of $x^4-7x^3+9x^2+7x+k=0$ have a sum of 4.

**F04A24.** In $\triangle ABC$, $AB=13$, $AC=15$ and $BC=4$. Compute the distance from $A$ to the closer of the two intersection points of the perpendicular bisector of $\overline{BC}$ and the circumcircle of $\triangle ABC$.

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ANSWERS:
F04A19. **69375**
F04A20. **3**
F04A21. $\dfrac{40}{11}$
F04A22. $\dfrac{1}{4}$
F04A23. **6**
F04A24. $\sqrt{65}$

**F04A25.** Compute the number of zeros to the right of the right-most non-zero digit of $1000! = 1 \times 2 \times 3 \times 4 \times \ldots \times 1000$.

**F04A26.** Compute all real values of $\theta$ (in radians) $\left(0 \leq \theta < 2\pi\right)$ that satisfy:
$1 + 2\sin\theta + 2\cos\theta + \sin 2\theta + \cos 2\theta = 0$.

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**F04A27.** Compute all real numbers $x$ such that $\sqrt[3]{8+x} + \sqrt[3]{8-x} = 1$

**F04A28.** A cube of edge 4 has each of its faces divided into 16 squares of side 1 each. Compute how many paths of length 12 there are from one vertex of the cube to the opposite vertex of the cube that are contained exclusively on the edges of the small squares drawn on the surface of the cube.

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**F04A29.** Compute the number of different two letter "words" that can be made from the letters in the word "RAMANUJAN".

**F04A30.** Let $A$ denote the point $(1,2)$, and let $M$ and $N$ be the intersections of the graphs of $x^2 + y^2 = 9$ and $y = 2x - 4$. Compute $AM^2 + AN^2$.

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**ANSWERS:**
F04A25. **249**
F04A26. $\dfrac{3\pi}{4}, \pi, \dfrac{7\pi}{4}$
F04A27. $\pm 3\sqrt{21}$
F04A28. **2550**
F04A29. **32**
F04A30. **28**

**F04A1.** Join the vertices with the center to form 12 congruent isosceles triangles, each with two sides of length 4 and a vertex angle of 30°. The area of the dodecagon is

$$12\left(\frac{1}{2}(4)(4)\sin 30°\right) = 48.$$

**F04A2.** We have   (i) $x^2 - 2xy = 0$ and  (ii) $x^2 = y + 3$.
From (i), $x(x - 2y) = 0$, so $x = 0$ or $x = 2y$. If $x = 0$, $\log_3 x$ is undefined.
If $x = 2y$, we have $4y^2 - y - 3 = 0$; $(4y + 3)(y - 1) = 0$ and
$y = -3/4, 1$ and $x = -3/2, 2$. Only the positive values will satisfy (ii).
Therefore the answer is **(2, 1)**.

**F04A3.** The minute and hour hands of the clock move at 360°/hour and 30°/hour, respectively, in the same direction. The angle between them changes at the rate of 330°/hour. Since the angle is 0° at noon, 24 minutes later it will be $\frac{24}{60}(330°) = \mathbf{132°}$.

**F04A4.** Let $A = \sqrt[3]{2+\sqrt{5}}$, $B = \sqrt[3]{2-\sqrt{5}}$, and $S = A + B$. Then
$(A+B)^3 = A^3 + B^3 + 3AB(A+B) = 4 - 3(A+B)$, so $S^3 + 3S - 4 = 0$. The cubic factors into $(S-1)(S^2 + S + 4) = 0$. Since $S^2 + S + 4 = 0$ has no real solutions, $S = 1$.

**F04A5.** One way of making \$81 is with three \$2 bills and fifteen \$5 bills. We could also trade in any two \$5 bills for five \$2 bills. We can perform this trade anywhere from 0 to 7 times, giving us **8** possibilities.

**F04A6.**

$$P = \left(\frac{2^1-1}{2^1}\times\frac{1}{2^2}\times\frac{1}{2^3}\times\ldots\times\frac{1}{2^{10}}\right) + \left(\frac{1}{2^1}\times\frac{2^2-1}{2^2}\times\frac{1}{2^3}\times\ldots\times\frac{1}{2^{10}}\right) + \left(\frac{1}{2^1}\times\frac{1}{2^2}\times\frac{2^3-1}{2^3}\times\ldots\times\frac{1}{2^{10}}\right) +$$

$$+\ldots+\left(\frac{1}{2^1}\times\frac{1}{2^2}\times\frac{1}{2^3}\times\ldots\times\frac{1}{2^9}\times\frac{2^{10}-1}{2^{10}}\right) = \frac{2^1-1+2^2-1+2^3-1+\ldots+2^{10}-1}{2^1\times 2^2\times 2^3\times\ldots\times 2^{10}} = \frac{2^{11}-12}{2^{55}}.$$

$$2^{11} - 12 = \mathbf{2036}.$$

**F04A7.** Each small block at a corner will have 3 painted faces. On each face the 9 central small blocks will have 1 painted face each. $8 + 6 \times 9 = \mathbf{62}$.

**F04A8.** Case 1: $x^2 - x - 3 \geq 0$, so $x^2 - 2x - 4 = 0$ and $x = 1 \pm \sqrt{5}$, only $1 + \sqrt{5}$ works. Case 2: $x^2 - x - 3 < 0$, so $x^2 - 2 = 0$, $x = \pm\sqrt{2}$, only $\sqrt{2}$ works. Ans: $\mathbf{1 + \sqrt{5}},\ \sqrt{2}$.

**F04A9.** $z^4 = \left(z^2\right)^2 = \left(2z - 2\right)^2 = 4z^2 - 8z + 4 = 4\left(z^2 - 2z\right) + 4 = 4(-2) + 4 = \mathbf{-4}$.

**F04A10.** The first 9 digits belong to the single-digit numbers. The next 180 digits belong to the two-digit numbers. The remaining $2004 - 9 - 180 = 1815$ digits belong to the first 605 three-digit numbers, in particular the last digit is the units digit of 704. Ans: **4**.

**F04A11.** $2003^2 - 1 = (2003 - 1)(2003 + 1)$, $2005^2 - 9 = (2005 - 3)(2005 + 3)$. Note that 2004 and 2008 have a greatest common factor of 4, so $4(2002) = \mathbf{8008}$ is the GCF.

**F04A12.** (see diagram)
$AG : GE = GE : GB$ (Since $\overline{EG}$ is an altitude from the vertex of the right angle of $\triangle AEB$, it splits that triangle into similar triangles $\triangle AEG$ and $\triangle EBG$) so $GE = 24$, and $BE = 30$.
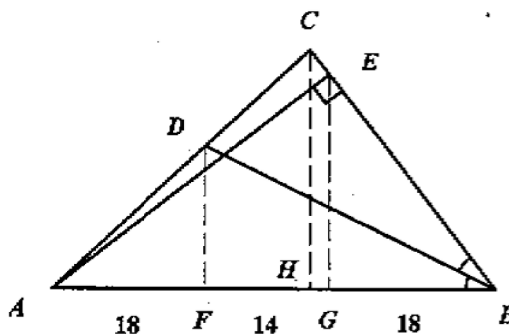$DF = BF \tan \angle ABD = 32 \tan \dfrac{\angle ABC}{2} = 16$
(using $\tan(X/2)$, as $\cos X = 3/5$ )
Let $h$ be the height from $C$ to $\overline{AB}$, and



$H$ be the foot of the altitude from $C$ to $\overline{AB}$. Then $AH + HB = 50$, and since $AH = CH \cot \angle CAB$, $BH = CH \cot \angle CBA$, we have $\dfrac{18h}{16} + \dfrac{18h}{24} = 50$, or $h = \dfrac{80}{3}$.

$(\triangle ABC) = \dfrac{1}{2} hAB = \dfrac{1}{2} \times \dfrac{80}{3} \times 50 = \dfrac{\mathbf{2000}}{\mathbf{3}}$

**F04A13.** Let $y = \log_8 x$. Then $y + \dfrac{1}{y} = \dfrac{10}{3}$, and so $y = 3$ or $y = \dfrac{1}{3}$. Ans: **2, 512.**

**F04A14.** The probability of an even number being picked is

$\dfrac{1}{2^2} + \dfrac{1}{2^4} + \dfrac{1}{2^6} + \ldots = \dfrac{\frac{1}{2^2}}{1 - \frac{1}{2^2}} = \dfrac{1}{3}$. The probability that an even multiple of 5 is picked is

$\dfrac{1}{2^{10}} + \dfrac{1}{2^{20}} + \dfrac{1}{2^{30}} + \ldots = \dfrac{\frac{1}{2^{10}}}{1 - \frac{1}{2^{10}}} = \dfrac{1}{1023}$.    Therefore the answer is $\dfrac{1}{3} - \dfrac{1}{1023} = \dfrac{340}{1023}$.

**F04A15.** We know that $r + s = 2$ and $rs = 2$. $r^2 + s^2 = (r + s)^2 - 2rs = 0$.

$r^4 + s^4 = (r^2 + s^2)^2 - 2(rs)^2 = -8$.

**F04A16.** (see diagram) Let $x$ and $y$ be the distances to
the shorter and longer chords, respectively. Note that
$\overline{AO}$ is the diagonal of a rectangle with sides $x$ and $y$,
so $x^2 + y^2 = 81$. However, since
$r^2 = 8^2 + x^2 = 10^2 + y^2$, we have

$2r^2 = 8^2 + 10^2 + 81 = 245$, so $r = \dfrac{7\sqrt{10}}{2}$.

**F04A17.** (all measurements in degrees)
$\cos 40 + \cos 80 = \cos(60 - 20) + \cos(60 + 20) = 2\cos 60 \cos 20 = \cos 20$.

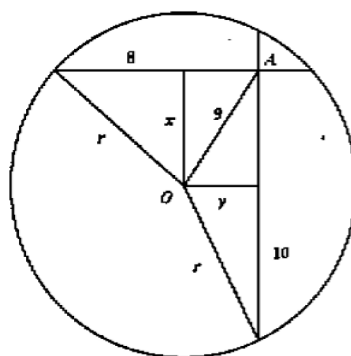**F04A18.** Note that $f(0) = 2f(0)^2$, so $f(0) = 0$ or $f(0) = \dfrac{1}{2}$.

CASE 1: If $f(0) = 0$, then for every $x$, $f(x) = 0^2 + f(x)^2$, so $f(x) = 0$ or $f(x) = 1$.
But the latter implies $f(2x) = 2f(x)^2 = 2$, which contradicts that for every $x$,
$f(x) \in \{0, 1\}$. So if $f(0) = 0$, then for all $x$, $f(x) = 0$.

CASE 2: If $f(0) = \dfrac{1}{2}$, then for every $x$, $f(x) = \dfrac{1}{4} + f(x)^2$, so $f(x) = \dfrac{1}{2}$

Ans: $0, \dfrac{1}{2}$.

**F04A19.** There are $5^3 = 125$ such numbers. They are symmetrically spaced around 555, so that must be their average. Their sum is $125 \times 555 = \mathbf{69375}$.

**F04A20.** Let $x = \sqrt{3 + 2\sqrt{2}} + \sqrt{6 - 4\sqrt{2}}$. Then

$x^2 = 3 + 2\sqrt{2} + 6 - 4\sqrt{2} + 2\sqrt{\left(3 + 2\sqrt{2}\right)(2)\left(3 - 2\sqrt{2}\right)} = 9 - 2\sqrt{2} + 2\sqrt{2} = 9$. Since $x > 0$,

$x = \mathbf{3}$.

**F04A21.** The combined rate of work is the sum of the three rates. $\dfrac{1}{8} + \dfrac{1}{12} + \dfrac{1}{15} = \dfrac{11}{40}$. The

pool will be filled in $\dfrac{\mathbf{40}}{\mathbf{11}}$ hours.

**F04A22.** Let $A = \dfrac{x^2 + 3x + 3}{(x+3)^2}$. Expressed as a quadratic in $x$ this yields

$x^2(A-1) + x(6A-3) + (9A-3) = 0$. In order for this quadratic to have a real solution in

$x$, we must have the discriminant nonnegative. $0 \le (6A-3)^2 - 4(A-1)(9A-3) = 12A - 3$,

so $A \ge \dfrac{1}{4}$. In fact, when $A = \dfrac{\mathbf{1}}{\mathbf{4}}$ solving for $x$ we obtain $x = -1$, so that is the minimum.

**F04A23.** $x^4 - 7x^3 + 9x^2 + 7x + k = 0$ has a sum of roots 7. If three of the roots have a sum of 4 then the remaining root must be 3. This means $x = 3$ must satisfy the equality, so $(3)^4 - 7(3)^3 + 9(3)^2 + 7(3) + k = 0$, and

$k = \mathbf{6}$.

**F04A24.** (see diagram) Let $O$ be the center of the circle. Let $M$ be the midpoint of $\overline{BC}$, and let $H$ be the foot of the altitude from $A$ to $\overline{BC}$. Let $\overline{XY}$ be the diameter of circle $O$ through $M$ as shown. By Heron's formula, the area of

$\triangle ABC$ is $\sqrt{(16)(1)(3)(12)} = 24$, so $AH = 12$

and $BH = 5$. Note that

$m\angle BOM = \dfrac{m\angle BOC}{2} = m\angle BAC$, so

$\dfrac{BM}{BO} = \sin \angle BOM = \sin \angle BAC = \dfrac{2(\triangle ABC)}{(13)(15)} = \dfrac{16}{65}$
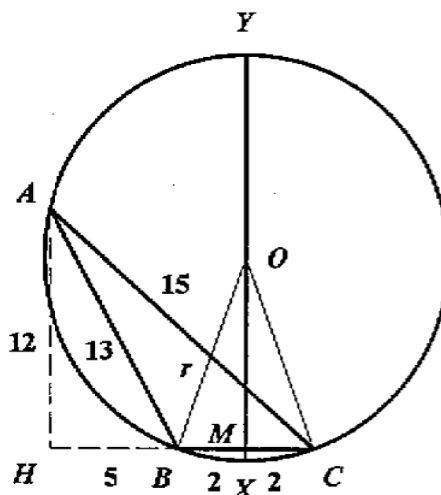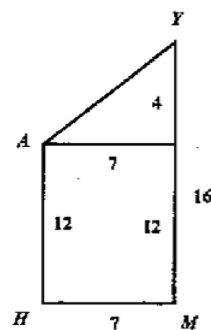


and since $BM = 2$ we have $r = BO = \dfrac{65}{8}$. By

Pythagoras, $MO = \dfrac{63}{8}$. Note that $MO = \dfrac{63}{8} < 12 = HA$, so $Y$ is

the intersection closer to $A$. $MY = MO + OY = \dfrac{63}{8} + \dfrac{65}{8} = 16$.

(see diagram) Finally, $YA = \sqrt{4^2 + 7^2} = \sqrt{\mathbf{65}}$.

**F04A25.** The number of zeros will be the power of 5 in the product. The product has 200 multiples of 5, of which 40 are multiples of 25, of which 8 are multiples of 125, of which 1 is a multiple of 625. $200+40+8+1=\mathbf{249}$.

**F04A26.** Using the double angle formulas we get:

$1+2\sin\theta+2\cos\theta+\left(2\sin\theta\cos\theta\right)+\left(2\cos^2\theta-1\right)=0$, or

$2\sin\theta+2\cos\theta+2\sin\theta\cos\theta+2\cos^2\theta=0$, which factors into

$2\left(\sin\theta+\cos\theta\right)\left(1+\cos\theta\right)=0$. Thus either $\sin\theta+\cos\theta=0$ or $1+\cos\theta=0$.

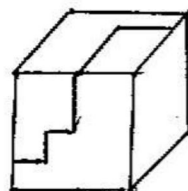These yield the solutions $\dfrac{3\pi}{4}$, $\pi$, **and** $\dfrac{7\pi}{4}$.

**F04A27.** Let $S=\sqrt[3]{8+x}+\sqrt[3]{8-x}$. Then

$S^3=1=8+x+8-x+3\left(\sqrt[3]{8+x}+\sqrt[3]{8-x}\right)\left(\sqrt[3]{(8+x)(8-x)}\right)=16+3\sqrt[3]{64-x^2}$, so

$3\sqrt[3]{64-x^2}=-15$, $64-x^2=-125$, so $x=\pm 3\sqrt{21}$.

**F04A28.** Any such path will lay entirely along two faces of the cubes (for example the path shown at right lies on the front and top faces.)

There are 6 such pairs of faces, and each pair yields $\dbinom{12}{4}$ paths.

However, we have overcounted - we need to subtract the paths which can be associated with more than one pair of faces. Those will be all the paths that include in the entirety an edge of the big cube - they traverse a face from corner to corner then follow an edge, or vice versa. There are 6 possible orientations of such face-edge pairs, each yielding $\dbinom{8}{4}$ paths. Note that $6\dbinom{12}{4}-6\dbinom{8}{4}=\mathbf{2550}$ includes each path exactly once.

**F04A29.** There are two "words" that can be made using the same letter and 30 that can be made using different letters, for a total of **32**.

**F04A30.** (see diagram) Let $P$ be the midpoint of $\overline{MN}$, and let $B$ be the foot of the perpendicular from $A$ to $\overline{MN}$. (It can be quickly calculated that it does in fact lie in the circle) Note that $\overline{OA}$ is parallel to $\overline{MN}$ so $OABP$ is a rectangle (important!). $AM^2=AB^2+BM^2$ and $AN^2=AB^2+NB^2$, so

$AM^2+AN^2=2AB^2+BM^2+NB^2=2OP^2+\left(PM-PB\right)^2+\left(PN+PB\right)^2$

$=2OP^2+PM^2+PB^2+PN^2+PB^2+2\left(PB\right)\left(PN-PM\right)$. Applying $PN=PM$, $PB=AO$,

and rearranging we have $AM^2+AN^2=2AO^2+2\left(OP^2+PM^2\right)=2\left(AO^2+OM^2\right)$. $OM$ is just the radius of the circle, and we can use the coordinates of $A$ to quickly find the other component. $AM^2+AN^2=2\left(AO^2+OM^2\right)=2(5+9)=\mathbf{28}$.