NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION | **CONTEST NUMBER ONE** | **FALL 2004**

PART I: 10 minutes | **NYCIML Contest One** | **Fall 2004**

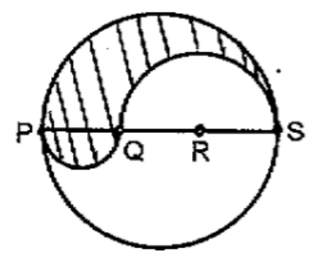
F04J1. Four couples (each consisting of a husband and a wife) go to a movie theater. They sit in a row of 8 seats. Compute the number of ways they can be seated, if each husband and wife has to sit next to each other.

F04J2. Compute the smallest real x for which $3^{2x+2}+1=3^{x+2}+3^{x}$.

PART II: 10 minutes | **NYCIML Contest One** | **Fall 2004**

F04J3. For all real numbers x, $f(x)=5x$ and $g(f(x))=x$. Compute $g(2004)$.

F04J4. Diameter $\overline{PS}$ of a circle of radius one is trisected by points Q and R. If $\overline{PQ}$ and $\overline{QS}$ are diameters of two smaller semicircles as shown, compute the area of the shaded portion of the figure.



PART III: 10 minutes | **NYCIML Contest One** | **Fall 2004**

F04J5. If 2^n is the largest power of 2 that divides $50!$ $(y!=1\times 2\times\ldots\times y)$, compute n.

F04J6. In right ΔABC, $c=5$ and $a+b=9$. Compute the area of ΔABC.

ANSWERS:

F04J1. **384**

F04J2. **-2**

F04J3. $\mathbf{\frac{2004}{5}}$ **or 400.8 or** $400\frac{4}{5}$

F04J4. $\mathbf{\frac{\pi}{3}}$

F04J5. **47**

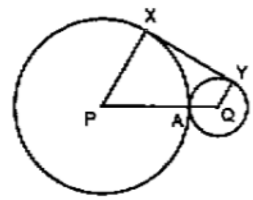
F04J6. $\mathbf{\frac{70}{9}}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
JUNIOR DIVISION CONTEST NUMBER TWO FALL 2004

PART I: 10 minutes **NYCIML Contest Two** **Fall 2004**

F04J7. A palindromic number is a positive integer that reads the same backwards and forwards (for example 12321 or 5665.) Compute the sum of all 3-digit palindromic numbers that are squares of 2-digit palindromic numbers.

F04J8. Circles P and Q are externally tangent at A. $\overline{XY}$ is a common external tangent of the circles, as shown. If the radii of the circles are 9 and 3, compute the area of quadrilateral $PQYX$.



PART II: 10 minutes **NYCIML Contest Two** **Fall 2004**

F04J9. Compute the surface area of a cube with a major diagonal (longest diagonal) of length $2\sqrt{3}$.

F04J10. Rhea drove from Emerald City to Old York City averaging 60 miles per hour. On the return trip, over the same route, she averaged only 50 mph. If the return trip took 24 minutes longer than the first trip, compute the distance between the two cities.

PART III: 10 minutes **NYCIML Contest Two** **Fall 2004**

F04J11. If the roots of $x^2+3x+7=0$ are p and q, and the roots of $x^2+ax+b=0$ are p^{-1} and q^{-1}, compute the ordered pair (a,b).

F04J12. w, x, y, and z are integers such that $\begin{cases} wx+yz=-1 \\ wy+xz=-1 \\ wz+xy=-1 \end{cases}$

If $w>1$, compute the ordered quadruple (w,x,y,z).

ANSWERS:

F04J7. **605**

F04J8. $\mathbf{36\sqrt{3}}$

F04J9. **24**

F04J10. **120**

F04J11. $\left(\frac{3}{7},\frac{1}{7}\right)$

F04J12. **(2,-1,-1,-1)**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION | **CONTEST NUMBER THREE** | **FALL 2004**

PART I: 10 minutes | **NYCIML Contest Three** | **Fall 2004**

F04J13. Compute the number of positive factors of 24^2.

F04J14. When Jan's son Danny is as old as Jan is now, Jan will be five times Danny's present age. When Jan and Danny's combined ages will be 50, Danny will be half of Jan's present age. Compute Jan's present age.

PART II: 10 minutes | **NYCIML Contest Three** | **Fall 2004**

F04J15. A room is in the shape of a cube with an edge of length a. A string is stretched taut from the midpoint of an edge on the floor of the room to the far most point on the ceiling. Express, in simplest form, the length of the string in terms of a.

F04J16. Compute the number of positive integers less than or equal to 101 that are multiples of perfect squares greater than one, but are not perfect squares themselves.

PART III: 10 minutes | **NYCIML Contest Three** | **Fall 2004**

F04J17. The expression $\sqrt{\dfrac{3^{10}+9^{8}}{9^{5}+3^{4}}}$ can be written as an integer N. Compute N.

F04J18. When $\dfrac{(2^3-1)(3^3-1)\ldots(100^3-1)}{(2^3+1)(3^3+1)\ldots(100^3+1)}$ is simplified, it can be written as $\dfrac{p}{q}$ where p and q are relatively prime integers. Compute the ordered pair (p,q).

ANSWERS:

F04J13. **21**
F04J14. **30**
F04J15. $\mathbf{\dfrac{3a}{2}}$ or equivalent
F04J16. **30**
F04J17. **27**
F04J18. **(3367,5050)**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER ONE SOLUTIONS** **FALL 2004**

F04J1. The four couples can be arranged in $4! = 24$ ways, and within each couple you have a choice between two possible seating orders for another $2^4 = 16$ possibilities. $24 \times 16 = \mathbf{384}$.

F04J2. Let $a = 3^x$. Then $9a^2 + 1 = 9a + a$, so $9a^2 - 10a + 1 = (9a-1)(a-1) = 0$. $a = \frac{1}{9}$ or $a = 1$, so $x = -2$ or $x = 0$. Of the two solutions, **-2** is smaller.

F04J3. $g(5x) = x$, so letting $y = 5x$ we have $g(y) = \frac{y}{5}$.

$$g(2004) = \frac{\mathbf{2004}}{\mathbf{5}} = \mathbf{400.8} = 400\frac{4}{5}$$

F04J4. $PQ = QR = RS = \frac{2}{3}$. The semicircle on diameter $\overline{PQ}$ has area $\frac{1}{2}\pi\left(\frac{1}{3}\right)^2 = \frac{\pi}{18}$. Likewise, the semicircle on diameter $\overline{QS}$ has area $\frac{4\pi}{18}$, and a semicircle on diameter $\overline{PS}$ has area $\frac{9\pi}{18}$. The area of the shaded region is $\frac{\pi}{18} + \frac{9\pi}{18} - \frac{4\pi}{18} = \frac{\boldsymbol{\pi}}{\mathbf{3}}$

F04J5. Of the first fifty integers, 25 are divisible by 2, of which 12 are divisible by 2^2, of which 6 are divisible by 2^3, of which 3 are divisible by 2^4, of which 1 is divisible by 2^5. In total this gives $25 + 12 + 6 + 3 + 1 = \mathbf{47}$ factors of 2.

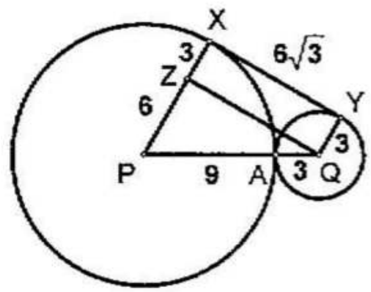
F04J6. Note that if $c = 5$ is the length of the hypotenuse then the sum of the legs would be the biggest if the triangle was a right isosceles triangle, yielding $\frac{5}{\sqrt{2}} + \frac{5}{\sqrt{2}} = 5\sqrt{2} < 9$. Therefore $c = 5$ is the length of a leg of the right triangle. Let x be the length of the other leg. Then the length of the hypotenuse is $9 - x$, and $(9-x)^2 = x^2 + 5^2$, so $x^2 - 18x + 81 = x^2 + 25$ and $18x = 56$, and $x = \frac{28}{9}$. The area is $\frac{1}{2}(5)\left(\frac{28}{9}\right) = \frac{\mathbf{70}}{\mathbf{9}}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

JUNIOR DIVISION **CONTEST NUMBER TWO** **FALL 2004**

SOLUTIONS

F04J7. We only need to consider $11^2 = 121$ and $22^2 = 484$, because $33^2 > 999$ is a 4-digit number. $121 + 484 = \mathbf{605}$



F04J8. (see diagram) Let Z be the foot of the perpendicular from Q to $\overline{XP}$. $ZXYQ$ is a rectangle, so $PZ = 9 - 3 = 6$, and since the hypotenuse of right ΔPQZ is $9 + 3 = 12$, $QZ = YX = 6\sqrt{3}$. The area of trapezoid $PQYX$ is $\frac{3+9}{2}\left(6\sqrt{3}\right) = \mathbf{36\sqrt{3}}$.

F04J9. In a cube of side a the major diagonal has length $a\sqrt{3}$. This means the cube has edges of length 2, and the surface area is $6\left(2^2\right) = \mathbf{24}$

F04J10. Let x = the distance between the cities. The time going was $\frac{x}{60}$, and the time returning was $\frac{x}{50}$. The given information tells us that $\frac{x}{50} - \frac{x}{60} = \frac{2}{5}$ hours. This yields $6x - 5x = 120$, or $x = \mathbf{120}$ miles.

F04J11. We know that $p + q = -3$ and $pq = 7$. Then $a = -\left(p^{-1} + q^{-1}\right) = -\frac{p+q}{pq} = \frac{3}{7}$, and $b = p^{-1}q^{-1} = \frac{1}{pq} = \frac{1}{7}$. The ordered pair is $\left(\mathbf{\frac{3}{7}, \frac{1}{7}}\right)$.

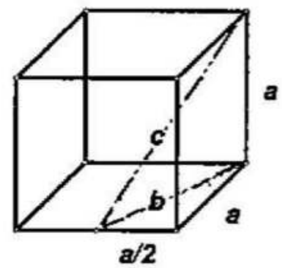
F04J12. Subtracting the first two equations gives $w(x-y) + z(y-x) = 0$ so that $(w-z)(x-y) = 0$. This means that either $w = z$ or $x = y$.

If $w = z$, the second equation gives $w(y+x) = -1$, which is impossible if $w > 1$.

If $x = y$, the last two equations become $x(w+z) = -1$ and $wz + x^2 = -1$. Since all four numbers are integers, we have $x = \pm 1$, so $x^2 = 1$ and $wz = -2$. Since $w > 1$, we have $w = 2$, $z = -1$. $x(w+z) = -1$ yields $x = y = -1$ and the ordered quadruple is $\mathbf{(2,-1,-1,-1)}$.

F04J13. $24^2 = \left(2^3 \times 3^1\right)^2 = 2^6 \times 3^2$. The number of positive factors is $(6+1)(2+1) = \mathbf{21}$

F04J14. Let J be Jan's age now, and D be Danny's age now. $J + (J - D) = 5D$, so $J = 3D$. Let T be the time from now when the sum of their ages is 50, so $(J+T)+(D+T)=50$, or $4D+2T=50$. Then $J = 2(D+T)$, or $3D = 2(D+T)$, and $D = 2T$. Substituting into $4D+2T=50$ we get $5D=50$, or $D=10$. $J = 3D = \mathbf{30}$



F04J15. (see diagram) By applying the Pythagorean Theorem twice we obtain $b^2 = a^2 + \left(\frac{a}{2}\right)^2$, and

$$c^2 = a^2 + b^2 = a^2 + a^2 + \left(\frac{a}{2}\right)^2 = \frac{9a^2}{4}, \text{ so } c = \mathbf{\frac{3a}{2}}$$

F04J16. There are 25 multiples of 4 less than 101. We cannot include 4, 16, 36, 64 or 100 since they are perfect squares. Thus there are **20** multiples to consider.

There are 11 multiples of 9 less than 101. We cannot include 9, 36 or 81 since they are perfect squares. We cannot include 72 since that was already included in the analysis for 4. This gives an additional **7** multiples to consider.

There are 4 multiples of 25 less than 101. We cannot include 25 or 100. Thus we have an additional **2** multiples to consider.

There are 2 multiples of 49 less than 101. We cannot include 49 itself. Thus there is only **1** additional multiple to consider.

The above totals 20 + 7 + 2 + 1 = **30**.

F04J17. $$\sqrt{\frac{3^{10}+9^8}{9^5+3^4}} = \sqrt{\frac{3^{10}+3^{16}}{3^{10}+3^4}} = \sqrt{\frac{3^{10}\left(3^6+1\right)}{3^4\left(3^6+1\right)}} = \sqrt{3^6} = 3^3 = \mathbf{27}$$

F04J18. Using the relationships:
$x^3+y^3=(x+y)\left(x^2-xy+y^2\right)$ *and* $x^3-y^3=(x-y)\left(x^2+xy+y^2\right)$, the given expression becomes:

$$\frac{1\left(2^2+2+1\right)\cdot 2\left(3^2+3+1\right)\cdot 3\left(4^2+4+1\right)\ldots 99\left(100^2+100+1\right)}{3\left(2^2-2+1\right)\cdot 4\left(3^2-3+1\right)\cdot 5\left(4^2-4+1\right)\ldots 101\left(100^2-100+1\right)} =$$

$$\frac{1\cdot 2\cdot 3\ldots\cdot 99\left(2^2+2+1\right)\left(3^2+3+1\right)\left(4^2+4+1\right)\ldots\left(100^2+100+1\right)}{3\cdot 4\cdot 5\ldots\cdot 101\left(2^2-2+1\right)\left(3^2-3+1\right)\left(4^2-4+1\right)\ldots\left(100^2-100+1\right)} =$$

$$\frac{2}{100\cdot 101}\cdot\left(\frac{7\cdot 13\cdot 21\cdot 31\cdot\ldots\left(100^2+100+1\right)}{3\cdot 7\cdot 13\cdot 21\cdot 31\cdot\ldots\left(100^2-100+1\right)}\right) = \frac{2}{3\cdot 100\cdot 101}\cdot\left(100^2+100+1\right) = \frac{2\cdot 10101}{3\cdot 100\cdot 101} = \frac{3367}{5050}$$

Thus the ordered pair is **(3367, 5050).**