New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER ONE** **SPRING 2004**

PART I: 10 minutes **NYCIML Contest One** **Spring 2004**

S04B01. Express $\sqrt{(365)^2-(364)^2}$ as a positive integer.

S04B02. The first three terms of a geometric progression are $x+1$, $2x$, and $2x+3$ in that order. Compute all possible values of the fourth term.

PART II: 10 minutes **NYCIML Contest One** **Spring 2004**

S04B03. If $8^{x+3}=8^x+1022$, compute x.

S04B04. Two intersecting circles whose radii are 3 and 7 have their centers 8 units apart. Compute the distance between the points of tangency of their common external tangent.

PART III: 10 minutes **NYCIML Contest One** **Spring 2004**

S04B05. If $[x]$ represents the greatest integer less than or equal to x, solve for x: $x[x]=50$

S04B06. Five pennies, five nickels and five dimes are in a box. Three coins are drawn at random without replacement. Compute the probability that the total value of these coins is less than $15¢$.

ANSWERS

1. 27
2. $-4,\ 13\frac{1}{2}$
3. $\frac{1}{3}$
4. $4\sqrt{3}$
5. $\frac{50}{7}$
6. $\frac{32}{91}$

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SENIOR B DIVISION **CONTEST NUMBER TWO** **SPRING 2004**

PART I: 10 minutes **NYCIML Contest Two** **Spring 2004**

S04B07. If $(5,-2)$, $(1,3)$, and $(-4,K)$ are collinear, compute K.

S04B08. If two roots of $x^3+px^2+q=0$ are 1 and 2, compute the third root.

PART II: 10 minutes **NYCIML Contest Two** **Spring 2004**

S04B09. Compute the area of the triangle with integral sides and perimeter 8.

S04B10. If the first three terms of a geometric progression are 3, $\sqrt[3]{9}$, and $\sqrt[9]{27}$, compute the fourth term.

PART III: 10 minutes **NYCIML Contest Two** **Spring 2004**

S04B11. If $\log_{10} x = 3 - 3\log_{10} 2$, compute x.

S04B12. In a circle, a diameter is drawn. One square is inscribed in the circle, and another square is inscribed in the semicircle. Compute the ratio of the area of the smaller square to the larger square.

ANSWERS

7. $\frac{37}{4}$

8. $-\frac{2}{3}$

9. $2\sqrt{2}$

10. 1

11. 125

12. $\frac{2}{5}$

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SENIOR B DIVISION **CONTEST NUMBER THREE** **SPRING 2004**

PART I: 10 minutes **NYCIML Contest Three** **Spring 2004**

S04B13. In an arithmetic progression, three times the third term equals six times the sixth term. Compute the ninth term.

S04B14. Compute the units digit of 7^{2004}.

PART II: 10 minutes **NYCIML Contest Three** **Spring 2004**

S04B15. In a fruit store, Darla buys 5 apples, 8 pears and 11 oranges, and pays \$2.63. Carla buys 3 apples, 5 pears, and 7 oranges and pays \$1.65. If Marla buys 1 apple, 1 pear and 1 orange, how much will she pay?

S04B16. In a circle of radius 10, two parallel chords are drawn on opposite sides of the center, each five units from the center. Compute the area of the region within the circle between the two chords.

PART III: 10 minutes **NYCIML Contest Three** **Spring 2004**

S04B17. Express $\frac{1-\cos(2x)}{\sin(2x)}$ as a single trigonometric function.

S04B18. In a triangle, two sides have lengths 10 and 12. The medians to these two sides are perpendicular to each other. Compute the length of the third side.

ANSWERS

13. 0

14. 1

15. 31¢

16. $\frac{100\pi}{3}+50\sqrt{3}$

17. $\tan x$

18. $\frac{2\sqrt{305}}{5}$

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER FOUR** **SPRING 2004**

PART I: 10 minutes **NYCIML Contest Four** **Spring 2004**

S04B19. Solve for all values of x:
$\sqrt{x^2+6x+9}=x+3$

S04B20. A regular octagon is inscribed in a circle with radius 12. Compute the area of the octagon.

PART II: 10 minutes **NYCIML Contest Four** **Spring 2004**

S04B21. Compute the sum of the infinite series
$.6-.06+.006-.0006+\ \ldots$.

S04B22. Compute the two roots of:
$(x-\frac{1}{4})(x-\frac{1}{4})+(x-\frac{1}{4})(x-\frac{1}{8})=0$

PART III: 10 minutes **NYCIML Contest Four** **Spring 2004**

S04B23. Compute the value of:

$$\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}}$$

S04B24. Compute the largest integral value of x for which 12^x is a factor of 20!

ANSWERS

19. $x \geq -3$
20. $288\sqrt{2}$
21. $\frac{6}{11}$
22. $\frac{1}{4}, \frac{3}{16}$
23. $-1+\sqrt{2}$
24. 8

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **SPRING 2004**

PART I: 10 minutes **NYCIML Contest Five** **Spring 2004**

S04B25. A $9\times9\times9$ cube is painted on all sides and then cut into $1\times1\times1$ cubes. Compute the number of cubes that have no paint on them.

S04B26. In a group of 4-legged dogs and 2-legged men, the total number of legs is 22 more than twice the number of heads. Compute the number of dogs.

PART II: 10 minutes **NYCIML Contest Five** **Spring 2004**

S04B27. If m and n are roots of $2x^2+3x+7=0$, compute m^3n+mn^3.

S04B28. An old printing press can print a newspaper in 8 hours. A new press can print it in 7 hours. 3 old and 2 new machines have been working for an hour when one old press breaks down. Compute the number of **minutes** it will take the other four machines to finish the job.

PART III: 10 minutes **NYCIML Contest Five** **Spring 2004**

S04B29. If $\left(a+b\sqrt{3}\right)^2=84+30\sqrt{3}$, compute all possible ordered pairs (a,b), where a and b are rational numbers.

S04B30. Compute the four prime factors of 1,000,027.

ANSWERS

25. **343**

26. **11**

27. $-\dfrac{133}{8}$

28. **38**

29. $(3,5),(-3,-5)$

30. **7, 19, 73, 103**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER ONE** **Spring 2004**

SOLUTIONS

S04B1.

$$\sqrt{365^2-364^2}$$
$$=\sqrt{(365+364)(365-364)}$$
$$=\sqrt{729}$$
$$=27.$$

S04B2. $\frac{x+1}{2x}=\frac{2x}{2x+3}$

$$2x^2+5x+3=4x^2$$
$$x=-\frac{1}{2}, x=3$$

If $x=3$, the first three terms are 4, 6, 9 and the fourth term is $13\frac{1}{2}$.

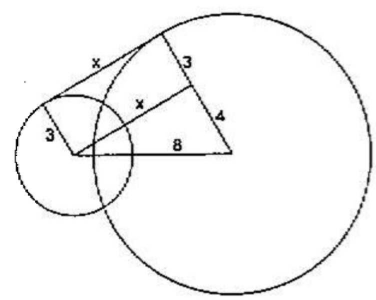
If $x=-\frac{1}{2}$, the first three terms are $\frac{1}{2}$, -1, 2 and the fourth term is -4.

S04B3. $8^3 8^x = 8^x + 1022$

$$511\cdot 8^x = 1022$$
$$8^x = 2$$
$$x=\frac{1}{3}$$

S04B4. $x^2+4^2=8^2$

$$x=\sqrt{48}=4\sqrt{3}$$



S04B5. x must be between 7 and 8. Therefore $[x]=7$. $7x=50$ and $x=\frac{50}{7}$.

S04B6. The number of ways 3 coins can be drawn is ${}_{15}C_3=455$. The ways that the value will be less than 15¢ is 3 pennies - ${}_5C_3$, 2 pennies and another coin - ${}_5C_2\cdot 10$, or 1 penny and 2 nickels - $5\cdot {}_5C_2$.

$$\frac{10+10\cdot 10+5\cdot 10}{455}=\frac{160}{455}=\frac{32}{91}.$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER TWO** **Spring 2004**

SOLUTIONS

S04B7. The slope between any 2 points must be the same.

$$\frac{K-3}{-5}=-\frac{5}{4}$$

$$K=\frac{37}{4}$$

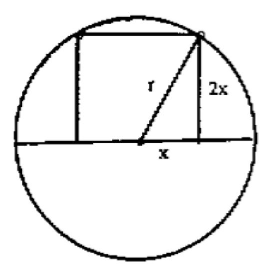
S04B8. Since the coefficient of x is 0, the sum of the roots taken 2 at a time is 0. Let the third root be R. $1\cdot 2+2\cdot R+1\cdot R=0$, so $R=-\frac{2}{3}$.

S04B9. The only triangle possible is the isosceles triangle with sides 3, 3, 2. The altitude is $\sqrt{8}$, and the area is $\frac{1}{2}\cdot 2\cdot\sqrt{8}=\sqrt{8}=\mathbf{2\sqrt{2}}$

S04B10. The terms are 3^1, $3^{\frac{2}{3}}$, $3^{\frac{1}{3}}$. The next term is 3^0, or 1.

S04B11. $\log_{10}x=\log_{10}1000-\log_{10}8=\log_{10}\frac{1000}{8}=\log_{10}125\Rightarrow x=125.$

S04B12. The larger square has diagonal equal to the diameter of the circle $A=\frac{1}{2}(2r)^2=2r^2$. For the smaller square let the side = 2x. $x^2+(2x)^2=r^2$, $5x^2=r^2$. $A=4x^2=\frac{4}{5}r^2$. $\frac{\frac{4}{5}r^2}{2r^2}=\frac{2}{5}$.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER THREE** **Spring 2004**

SOLUTIONS

S04B13. The first term is n, the third term is $n + 2d$, the sixth term is $n + 5d$, and the ninth term is $n + 8d$.

$3(n + 2d) = 6(n + 5d)$

$3n + 24d = 0 \rightarrow n + 8d = 0$

S04B14. The units digit of the powers of 7 runs in cycles of 4: 7, 9, 3, 1. Since 2004 is a multiple of 4, the units digit is the fourth number in a cycle, or 1.

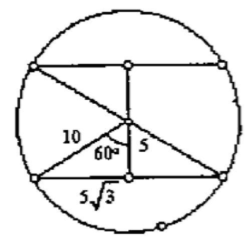
S04B15. $5A + 8P + 11R = 263$

$3A + 5P + 7R = 165$

Multiplying the first equation by 2, the second by 3 and subtracting, $A + P + R = 31¢$.

S04B16. The area consists of 2 sectors of 60° and two triangles with base $10\sqrt{3}$ and height 5.

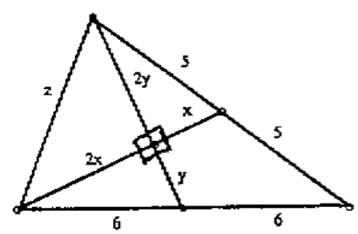
$$2 \cdot \frac{1}{6} \cdot 100\pi + 2 \cdot \frac{1}{2} \cdot 5 \cdot 10\sqrt{3} = \frac{100\pi}{3} + 50\sqrt{3}$$



S04B17. $\dfrac{1-\cos 2x}{\sin 2x} = \dfrac{1-(1-2\sin^2 x)}{2\sin x\cos x} = \dfrac{\sin x}{\cos x} = \tan x.$

S04B18. The medians intersect $\frac{2}{3}$ of the way to the other side.

$$x^2 + (2y)^2 = 25$$
$$(2x)^2 + y^2 = 36$$
$$5x^2 + 5y^2 = 61$$
$$z^2 = 4x^2 + 4y^2 = \frac{4}{5} \cdot 61 = \frac{244}{5}$$
$$z = \sqrt{\frac{244}{5}} = \frac{2\sqrt{61}}{\sqrt{5}} = \frac{2\sqrt{305}}{5}.$$



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER FOUR** **Spring 2004**

SOLUTIONS

S04B19. $\sqrt{x^2+6x+9}$ will equal $x+3$ if and only if $x+3 \geq 0$. $x \geq -3$.

S04B20. The central angle equals 45^0. There are 8 congruent isosceles triangles with legs 12 and included angle 45^0. $A = 8 \cdot \frac{1}{2} \cdot 12 \cdot 12 \cdot \sin(45^0) = 288\sqrt{2}$.

S04B21. This is an infinite geometric series with common ratio $-\frac{1}{10}$. $\dfrac{\frac{6}{10}}{1-\left(-\frac{1}{10}\right)} = \frac{6}{11}$.

S04B22. $\left(x-\frac{1}{4}\right)\left[x-\frac{1}{4}+x-\frac{1}{8}\right]=0$

$x=\frac{1}{4}$, $2x=\frac{3}{8}$ $x=\frac{3}{16}$.

S04B23. $x=\frac{1}{2+x}$, $x^2+2x-1=0$ $x=\frac{-2\pm\sqrt{8}}{2}$. Reject the negative answer, $x=-1+\sqrt{2}$.

S04B24. Since $12 = 2^2 \cdot 3$, we must find how many factors of 2 and 3 are in 20! Each multiple of 2 will give one 2. Each multiple of 4 will give 2 2's, etc. Likewise, each multiple of 3 will give one 3, each multiple of 9 will give 2 3's etc. $20! = 2^{18} \cdot 3^8 N$. 12^8 is the largest power of 12.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER FIVE Spring 2004

SOLUTIONS

S04B25. The inner cube of $7\times 7\times 7$ will remain unpainted. $7\times 7\times 7=343$.

S04B26. Let D = the number of dogs, M= the number of men. $4D+2M=22+2(M+D)$. $D=11$.

S04B27. $m^3n+mn^3=mn(m^2+n^2)=mn\left[(m+n)^2-2mn\right]$. $m+n=-\frac{3}{2}$, $mn=\frac{7}{2}$.

$$\frac{7}{2}\left[\left(\left(-\frac{3}{2}\right)^2-2\cdot\frac{7}{2}\right)\right]=\frac{7}{2}\left[\frac{9}{4}-7\right]=-\frac{133}{8}.$$

S04B28. In the first hour the process will complete $\frac{3}{8}+\frac{2}{7}$ of the job.

$\frac{3}{8}+\frac{2}{7}+\frac{2x}{8}+\frac{2x}{7}=1$, $x=\frac{19}{30}$ hours = 38 minutes.

S04B29. $\left(a+b\sqrt{3}\right)^2=a^2+2ab\sqrt{3}+3b^2=84+30\sqrt{3}$. Equating coefficients, $a^2+3b^2=84$
$2ab=30$, $ab=15$.
$(3,5),(3,-5)$.

S04B30. Using $a^3+b^3=(a+b)(a^2-ab+b^2)$, $1{,}000{,}027=(100)^3+3^3=$
$(100+3)\left(100^2-3\cdot 100+3^2\right)=103\cdot 9709=103\cdot 7\cdot 19\cdot 73.$