SENIOR A DIVISION | **CONTEST NUMBER ONE** | **SPRING 2004**

PART I: 10 minutes | **NYCIML Contest One** | **Spring 2004**

S04A1. If $\frac{2x}{x^2+1}+\frac{2x^2+2}{x}=5$, compute all real values of x.

S04A2. John, Paul, George, and Ringo bought 15 identical ties. Compute the number of ways they can divide the ties among themselves if each one gets at least one tie.

PART II: 10 minutes | **NYCIML Contest One** | **Spring 2004**

S04A3. Points X and Y are on the x-axis and y-axis, respectively, so that the slope of $\overline{XY}$ is 2. Let A be the foot of the perpendicular from the origin to $\overline{XY}$. Compute the ratio $XA:YA$.

S04A4. If x is the measure of an acute angle and $2-\tan x=\cot 2x$, compute $\sin x+\cos x$.

PART III: 10 minutes | **NYCIML Contest One** | **Spring 2004**

S04A5. Nine of the ten integers 2, 3, 4, ..., 11 are placed in the squares of a 3 by 3 grid, one number to a square, in such a way that the six sums of the numbers of each row and each column are identical. Let x be the number that is not among those placed. If x is not a prime, compute x.

S04A6. In circle O, chord $\overline{AB}$ of length 2 is parallel to diameter $\overline{CD}$. If $AC=3$, compute all possible values of the radius of the circle.

ANSWERS:

S04A1. **1**

S04A2. **364**

S04A3. **1:4 or 0.25**

S04A4. $\frac{\sqrt{6}}{2}$

S04A5. **8**

S04A6. $\frac{\sqrt{19}\pm 1}{2}$

PART I: 10 minutes **NYCIML Contest Two** **Spring 2004**

S04A7. If $a = \log 2$ and $b = \log 3$, express $\log_{81} 80$ in terms of a and b with no logarithms.

S04A8. Two circles are drawn in the plane. Lines l and k are each tangent to both circles with the points of tangency 24 units apart on line l and 66 units apart on line k. If the radius of one circle is 15, compute the radius of the other circle.

PART II: 10 minutes **NYCIML Contest Two** **Spring 2004**

S04A9. Linda has four dimes, three of which are fair coins. The fourth dime has heads on both sides. She picks a coin at random and flips it, obtaining a head. Compute the probability that the coin is a fair dime.

S04A10. In circle O, perpendicular chords $\overline{AB}$ and $\overline{BC}$ have lengths 10 and 24, respectively. Circle Q is tangent to $\overline{AB}$, $\overline{BC}$ and circle O. Compute the radius of circle Q.

PART III: 10 minutes **NYCIML Contest Two** **Spring 2004**

S04A11. In ΔABC $\tan A = 2$ and $\tan B = 4$. Compute $\tan C$.

S04A12. Consider all of the 5-digit natural numbers that can be formed using the digits 6, 7, 8, 9, and 0, using each digit exactly once. Compute the sum of these numbers.

ANSWERS:

S04A7. $\mathbf{\frac{1+3a}{4b}}$

S04A8. **63**

S04A9. $\mathbf{\frac{3}{5}}$

S04A10. **8**

S04A11. $\mathbf{\frac{6}{7}}$

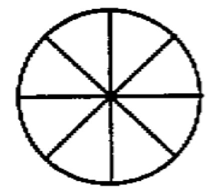
S04A12. **7799940**

SENIOR A DIVISION　　CONTEST NUMBER THREE　　SPRING 2004

PART I: 10 minutes　　**NYCIML Contest Three**　　**Spring 2004**

S04A13. If 8,731,3a4 is divisible by 24, compute a.

S04A14. A pizza is cut into 8 slices as shown. Five randomly chosen slices are eaten. Compute the probability that none of the 3 remaining slices share a side.



PART II: 10 minutes　　**NYCIML Contest Three**　　**Spring 2004**

S04A15. The sum of the first 10 terms of an arithmetic sequence is 204. The sum of the next 10 terms is 2004. Compute the first term of the sequence.

S04A16. In ΔABC, $m\angle B = 90°$. Point P is chosen outside of ΔABC so that $PA = 9$, $PB = 6$, $PC = 4$, and $\angle PAB \cong \angle PBC$. Compute the area of ΔABC.

PART III: 10 minutes　　**NYCIML Contest Three**　　**Spring 2004**

S04A17. In isosceles ΔXYZ, $XY = XZ$. Point W is chosen on the base so that $XW = YW = 10$ and the distance from W to $\overline{XY}$ is 6. Compute WZ.

S04A18. Compute the number of integers between 10 and 1000, whose digits add up to 9.

ANSWERS:

S04A13. **4**

S04A14. $\mathbf{\frac{2}{7}}$

S04A15. **-60.6 or –303/5**

S04A16. **15**

S04A17. **15.6 or 78/5**

S04A18. **54**

SENIOR A DIVISION | **CONTEST NUMBER FOUR** | **SPRING 2004**

PART I: 10 minutes | **NYCIML Contest Four** | **Spring 2004**

S04A19. Compute the number of lattice points (points with integer coordinates) in the interior of the triangle with vertices (0, 0), (12, 0), and (5, 6).

S04A20. In ΔABC, $m\angle A = 3m\angle B$. If BC = 6 and AC = 4, compute AB.

PART II: 10 minutes | **NYCIML Contest Four** | **Spring 2004**

S04A21. Compute the length of the altitude to the base of an isosceles triangle given that the base has length $3\sqrt{10}$, and the altitude to either leg has length 9.

S04A22. Compute all integers x for which $x^4 + 4x^3 - 8x + 1$ is a perfect square.

PART III: 10 minutes | **NYCIML Contest Four** | **Spring 2004**

S04A23. Andy and Roger each roll a regular 6-sided fair die. Compute the probability that Andy's result is an integral multiple of Roger's result.

S04A24.

$$x + yz = 1$$
$$y + xz = -1$$
$$z + xy = 1$$

If x, y, and z are complex numbers, compute all possible values of xyz.

ANSWERS:

S04A19. **30**

S04A20. $\sqrt{10}$

S04A21. $\frac{9\sqrt{10}}{2}$

S04A22. **0, -2**

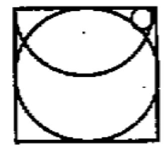
S04A23. $\frac{7}{18}$

S04A24. **-2, 1**

SENIOR A DIVISION **CONTEST NUMBER FIVE** **SPRING 2004**
PART I: 10 minutes **NYCIML Contest Five** **Spring 2004**

S04A25. How many ordered pairs of positive integers (a,b) exist such that the greatest common divisor of a and b is 11, and the least common multiple of a and b is 2002.

S04A26. A circle and a semicircle, each of radius 150, are inscribed into a square as shown. Compute the radius of the small circle that is tangent to the large circle, the semicircle, and the square.



PART II: 10 minutes **NYCIML Contest Five** **Spring 2004**

S04A27. a is chosen so that $x^2+4x+a=0$ and $2x^2+ax+2=0$ each have two distinct real roots. Compute all real numbers a for which the absolute value of the difference of the roots is the same for both of the quadratic equations.

S04A28. Seven girls and three boys are to be arranged in a line so that each boy is to have a girl on either side. Compute the number of ways in which this is possible.

PART III: 10 minutes **NYCIML Contest Five** **Spring 2004**

S04A29. Ming ordered a (circular) pizza, and now he wants to cut it into 10 pieces with n cuts (each cut is along a plane perpendicular to the top of the pizza and contains a chord of the circular pizza top.) Compute the number of different values of n that will make this possible.

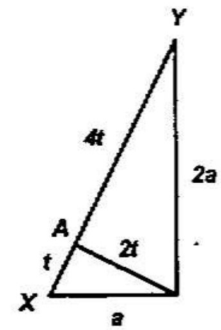
S04A30. Compute the number of 7-digit integers, containing no digits other than 1 or 2, in which neither three or more 1's nor four or more 2's appear in a row.

ANSWERS:
S04A25. **8**
S04A26. **25**
S04A27. **-20**
S04A28. **604800**
S04A29. **6**
S04A30. **63**

S04A1. Let $t = \dfrac{x}{x^2+1}$, then $t + \dfrac{1}{t} = \dfrac{5}{2}$. $t = 2$ or $t = \dfrac{1}{2}$. $\dfrac{x}{x^2+1} = 2$ has no solutions. $\dfrac{x}{x^2+1} = \dfrac{1}{2}$ has one solution $x = 1$.

S04A2. The problem corresponds to determining the number of positive integral solutions to $x_1 + x_2 + x_3 + x_4 = 15$, which is $\dbinom{14}{3} = 364$.

S04A3. By similar triangles, (see diagram) the ratio is $\dfrac{1}{4}$ or 1:4.



S04A4. $4 - 2\tan x = 2\cot 2x = \dfrac{1-\tan^2 x}{\tan x} = \cot x - \tan x$

$4 = \cot x + \tan x = \dfrac{1}{\sin x \cos x} \rightarrow \sin x \cos x = \dfrac{1}{4}$

$(\sin x + \cos x)^2 = 2\sin x \cos x + 1 = \dfrac{1}{2} + 1 = \dfrac{3}{2}$

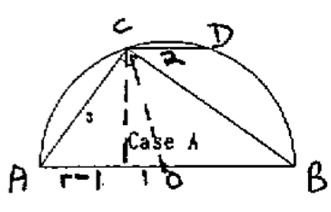
$\sin x + \cos x = \dfrac{\sqrt{6}}{2}$

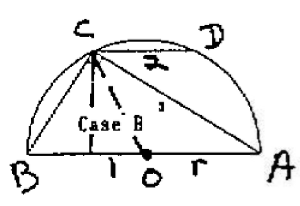
S04A5. Let the sum along a column be s, then the sum of the 9 numbers must be $3s$ and similarly the sum along each row must also be s. Since 2+3+…+11 = 65, $3s$ must be one of the 4 multiples of 3 between 65-11 and 65-2 inclusive. Of the 4 numbers that can be excluded, only 8 is not prime, so **8** is the answer.

2	11	6
7	3	9
10	5	4

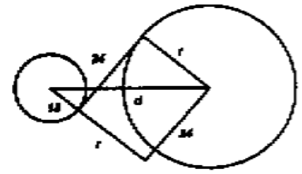
S04A6. Let the radius of the circle be r. In case A, the perpendicular distance between the two lines must be $\sqrt{9-(r-1)^2}$. Since $r^2 = 9-(r-1)^2+1 = 9+2r-r^2$, $r = \dfrac{\sqrt{19}+1}{2}$.

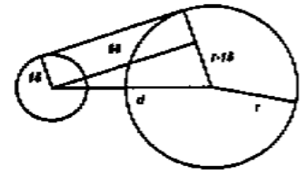
In case B, the perpendicular distance between the two lines must be $\sqrt{9-(r+1)^2}$. Since $r^2 = 9-(r+1)^2+1 = 9-2r-r^2$, $r = \dfrac{\sqrt{19}-1}{2}$. The possible answers are $r = \dfrac{\sqrt{19}\pm 1}{2}$.





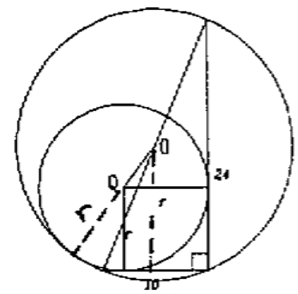
S04A7. $\log_{81} 80 = \dfrac{\log 10 + 3\log 2}{4\log 3} = \dfrac{1+3a}{4b}$





S04A8. Let the radius of the other circle be r and d is the distance between the centers, then regardless of which circle is larger, the equation is $(r+15)^2 + 24^2 = d^2 = (r-15)^2 + 66^2$. Solving, this simplifies to a linear equation and $r = 63$.

S04A9. Of the 5 existing heads, 3 lie on a fair coin. Thus the conditional probability is **3/5**.

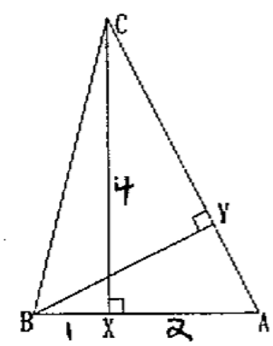


S04A10. The length $(OQ)^2 = (13-r)^2 = (12-r)^2 + (r-5)^2$. This leaves $r = 0$ or $r = \mathbf{8}$.

S04A11.

Soln A) Build one such triangle with $CX = 4$. It follows that the area of ΔABC is 6, $BC = \sqrt{17}$, and $AC = 2\sqrt{5}$.

Area $= 6 = \frac{1}{2}(2\sqrt{5})(BY)$. $BY = \dfrac{6\sqrt{5}}{5}$. By the Pythagorean Theorem, $CY = \dfrac{7\sqrt{5}}{5}$. In triangle CBY, $\tan C = \mathbf{\dfrac{6}{7}}$.



Soln B) In a triangle,
$(\tan A)(\tan B)(\tan C) = \tan A + \tan B + \tan C$,
so $2(4)(x) = 2 + 4 + x$, and $x = 6/7$

Soln C) $\tan C = \dfrac{\tan ACX + \tan BCX}{1 - \tan ACX \tan BCX} = \dfrac{\frac{1}{2} + \frac{1}{4}}{1 - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)} = \dfrac{4+2}{8-1} = \dfrac{6}{7}$

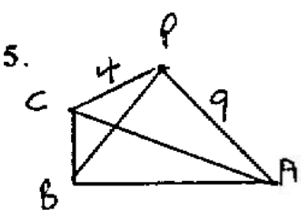
S04A12. If the leftmost digit was allowed to be 0, then the average digit is 6 and there are 120 such numbers, totaling to $120 \cdot 6 \cdot 11111 = 7999920$. The numbers whose leftmost digit is 0 can be considered 4 digit numbers consisting of 6, 7, 8, or 9, with an average digit of 7.5. There are 24 such numbers, totaling to $24 \cdot 7.5 \cdot 1111 = 199980$. $7999920 - 199980 =$ **7799940.**

S04A13. The number must be divisible by 24, that is, divisible by 3 x 8, hence must be divisible by both 8 and 3. For it to be divisible by 3, a must be either 1, 4, or 7. For the number to be divisible by 8, the number $3a4$ must be divisible by 8. only $a = 4$ works.

S04A14. The total number of combinations of 3 pieces is $\binom{8}{3} = 56$. The number of combinations that contain exactly 2 pieces stuck together is $8 \cdot 4$, since there are 8 places for the lone piece to be and 4 places for the double piece to be after the lone piece is chosen. The number of combinations that contain 3 pieces stuck together is 8. So $\frac{56-32-8}{56} = \frac{2}{7}$.

S04A15. Let a be the first term and d be the common difference. The tenth term is $a+9d$. The average of the first ten terms is the average of the first and last term, or $a+4.5d = 20.4$ The eleventh and the twentieth terms are $a+10d$ and $a+19d$. The average of the second ten terms is the average of the first and last of those, or $a+14.5d = 200.4$. Solving, we find $d = 18$, and so $a = \mathbf{-60.6}$.

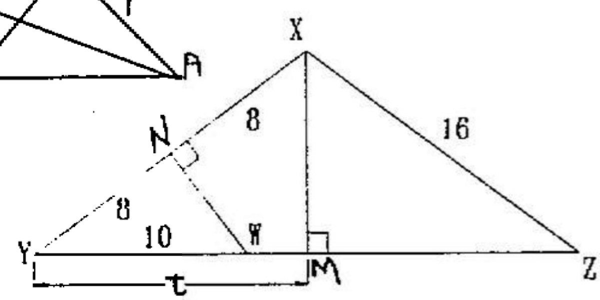
S04A16. $\angle ABP$ is complementary to $\angle PBC$, so also to $\angle PAB$. Thus $\angle APB$ is a right angle, and $AB = 3\sqrt{13}$. $\cos\angle PBC = \cos\angle PAB = \frac{9}{3\sqrt{13}} = \frac{3\sqrt{13}}{13}$. Applying Law of Cosines to ΔPBC, we find $16 = 36 + BC^2 - 2(6)(BC)\frac{3\sqrt{13}}{13}$. We get two roots, $2\sqrt{13}$ and $\frac{10\sqrt{13}}{13}$, but only for the second does point P lie outside of ΔABC. The area of ΔABC is $\frac{1}{2}\left(3\sqrt{13}\right)\left(\frac{10\sqrt{13}}{13}\right) = \mathbf{15}$.



S04A17. $\Delta YNW \cong \Delta YMX$

Therefore $t = \frac{8 \cdot 16}{10} = \frac{64}{5}$.

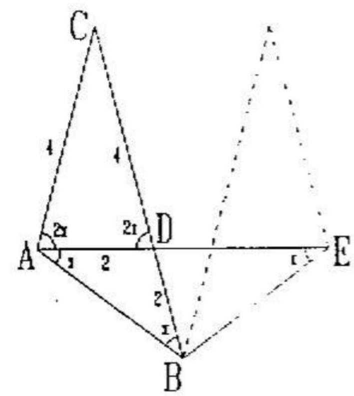
$WZ = 2t - 10 = \frac{128}{5} - \frac{50}{5} = \frac{78}{5}$



S04A18. Solution 1: There are 9 such two-digit numbers, 18, 27, …, 90. If the hundreds digit is 1, there are 9 numbers, if it's 2 there are 8 numbers, …, if the hundreds digit is 9 there is 1 number. $9 + (9+8+...+1) = \mathbf{54}$

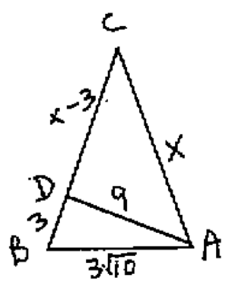
Solution 2: The problem simplifies to distributing 9 units among 3 containers. You can do that in $\binom{9+3-1}{3-1} = \binom{11}{2} = 55$ ways. This includes the number 9, so the answer is $55 - 1 = \mathbf{54}$.

S04A19. The area of the triangle is $\frac{|6\cdot 12-0\cdot 5|}{2}=36$. There are 13 points on the segment from the (0,0) to (12,0), 2 points on the segment from (0,0) to (5,6), and 2 points on the segment from (12,0) to (5,6), totaling to 14 points on the edge of the triangle. By Pick's Formula, 36 = I + E/2 -1 = I + 7 – 1. Thus I = 30. there are 30 points in the interior of the triangle.



S04A20. Let the measure of angle B be x. Locate D on segment BC such that the measure of angle BAD is also x. Then angle ADC must be $2x$ since it is the supplement to angle ADB. Triangle CAD is thus isosceles. Extend AD through D to E such that angle BED measures x. Drawing a line parallel to AC through B and a line parallel to BC through E, similar triangles form and the length AE can be determined to be 5. Since triangle ADB is similar to triangle ABE, the length $AB=\sqrt{10}$.

S04A21. Let the leg of the triangle be x, then the distance BD is 3 by the Pythagorean theorem, so $x^2=(x-3)^2+9^2$. x is thus 15. The altitude to the base of the triangle is $\frac{15\cdot 9}{3\sqrt{10}}=\frac{9\sqrt{10}}{2}$.



S04A22. . $(x^2+2x)^2=x^4+4x^3+4x^2$ matches the polynomial down to the 3rd power. $(x^2+2x-2)^2=x^4+4x^3-8x^2+4$ will match the polynomial down to the 2nd and coincidentally the 1st power too. In other words, $x^4+4x^3-8x^2+1=(x^2+2x-2)^2-3$. 3 can be the difference of squares only when $x^2+2x-2=\pm 2$, or when $\mathbf{x=0}$ **or** $\mathbf{x=-2}$.

S04A23. $\frac{1}{6}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right)=\frac{7}{18}$

S04A24. Let $p=xyz$, then

$yz=1-x$

$xz=-1-y$

$xy=1-z$

$p^2=-(x-1)(y+1)(z-1)=-p+xy+yz-xz+x-y+z-1=-p+(xy+z)+(yz+x)-(xz+y)-1$

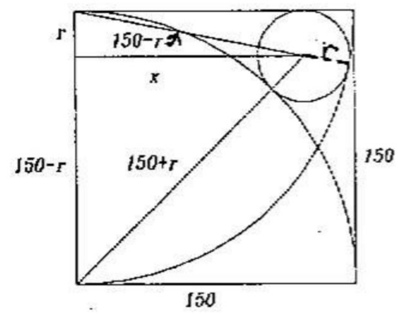
$p^2=-p+1+1+1-1=2-p$

$p=1$ or $p=-2$

SOLUTIONS

S04A25. Let $a = 11a_1$, $b = 11b_1$. Now for each of 2, 7, and 13, at only one of a_1 or b_1 contain it. Hence we have $2 \cdot 2 \cdot 2 = 8$ total possibilities for (a_1, b_1), and hence (a, b).

S04A26. $x^2 = (150 - r)^2 - r^2 = (150 + r)^2 - (150 - r)^2$
Solving for r yields a linear equation that reduces to $r = 25$



S04A27. The quadratic equation $x^2 + 4x + a = 0$ has as roots $\frac{-4 \pm \sqrt{16-4a}}{2}$, the absolute value of the difference of the roots is $\sqrt{16-4a}$, and the quadratic equation $2x^2 + ax + 2 = 0$ has roots $\frac{-a \pm \sqrt{a^2-16}}{4}$, the absolute value of the difference of the roots is $\frac{\sqrt{a^2-16}}{2}$. If the 2 differences are equal, we have $\sqrt{16-4a} = \frac{\sqrt{a^2-16}}{2}$, leading to $4(16-4a) = a^2 - 16$. Solving, we get $a = 4$ or $a = -20$. If $a = 4$, neither equation has distinct roots. Hence $a = -20$ is the only solution.

S04A28. Arrange the girls first – this can be done in 7! ways. Now we must place the boys between the girls. There are 6 such positions, and each can contain at most 1 boy. Hence, we have $6 \cdot 5 \cdot 4$ arrangements for the boys after we arranged the girls, for a total of $7! \cdot 6 \cdot 5 \cdot 4 = 604800$

S04A29. 4 is the smallest number of cuts needed. 9 is the largest possible. Anything in between also works. Thus there are a total of 6 possibilities.

S04A30. The number of 7 digit numbers made up of 1's and 2's without restrictions is $2^7 = 128$. Sort the numbers that contain 3 adjacent 1's by the starting position of the consecutive 1's, then there are a total of $2^4 + 2^3 + 2^3 + 2^3 + (2^3 - 1) = 47$ numbers that contain 3 adjacent 1's. Sort the numbers that contain 4 adjacent 2's in a similar manner, then there are $2^3 + 2^2 + 2^2 + 2^2 = 20$. There are 2 numbers that have both of these qualities. The total number thus tallies to $128 - 47 - 20 + 2 = 63$.