JUNIOR DIVISION **CONTEST NUMBER ONE** **SPRING 2004**

PART I: 10 minutes **NYCIML Contest One** **Spring 2004**

S04J1. Five men can plow a square field whose side is 60 feet in 4 hours. At the same rate of work, in how many hours could 10 men plow a square field whose side is 180 feet?

S04J2. In ΔABC, the median to side BC is perpendicular to the median to side AC. If $BC = 7$ and $AC = 6$, compute AB.

PART II: 10 minutes **NYCIML Contest One** **Spring 2004**

S04J3. In a regular octagon, all diagonals are drawn. Compute how many points lie on two or more diagonals.

S04J4. Alf can build a spaceship in 33 days. His rate is 40% faster than Willie's and 25% faster than Kate's. If all three of them work together, how long will it take them to build a spaceship?

PART III: 10 minutes **NYCIML Contest One** **Spring 2004**

S04J5. Kim always counts in base 8 and computes that she has 555 dollars. Jing always counts in base 9 and computes that he has 455 dollars. Compute and express in base 10 the difference between the larger and the smaller amounts.

S04J6. Through each vertex of ΔABC a line is drawn that is perpendicular to the angle bisector at that vertex. The three lines drawn intersect at points *X*, *Y*, and *Z*. If $AB = 3$, $BC = 4$, and $AC = 5$, then the perimeter of ΔXYZ can be written as $3\sqrt{p} + 4\sqrt{q} + 5\sqrt{r}$, where *p*, *q*, and *r* are integers. Compute the ordered triple (p, q, r).

ANSWERS:

S04J1. **18**

S04J2. $\sqrt{17}$

S04J3. **49**

S04J4. $\frac{660}{47}$ or $14\frac{2}{47}$

S04J5. **9**

S04J6. $(10,5,2)$

JUNIOR DIVISION **CONTEST NUMBER TWO** **SPRING 2004**

PART I: 10 minutes **NYCIML Contest Two** **Spring 2004**

S04J7. A glass is 1/3 filled with milk and an identical glass is 1/4 filled with milk. Each glass is now filled completely by adding water and both glasses are emptied into a pitcher. Half of the mixture is now poured back into one of the glasses. Compute the fractional part of the mixture in that glass that is milk.

S04J8. Compute the smallest positive integer greater than 2004 that gives a remainder of 4 when divided by 5, a remainder of 5 when divided by 6, and a remainder of 6 when divided by 7.

PART II: 10 minutes **NYCIML Contest Two** **Spring 2004**

S04J9. Jason bought 12 pens and 4 pencils for \$60. At the same price, he could buy 3 more pens for \$24 than pencils for \$30. Compute the price of a pen.

S04J10. In ΔXYZ $m\angle X = 45°$, $m\angle Y = 30°$, and M is the midpoint of $\overline{XY}$. N is chosen on $\overline{YZ}$ so that $\overline{MN}$ is the base of isosceles ΔMNY. If $MN = 1$, compute NZ.

PART III: 10 minutes **NYCIML Contest Two** **Spring 2004**

S04J11. Consider all 5-digit positive integers whose digits are the numbers 1, 2, 3, 4, and 5 (no repetition of digits). If these 5-digit positive integers are listed in increasing order, what is the 80^{th} entry on the list?

S04J12. Find the smallest real number x such that $|x-1|-2|x+3|+x+7=0$.

ANSWERS:

S04J7. **7/24**

S04J8. **2099**

S04J9. **3**

S04J10. $\dfrac{3\sqrt{2}-\sqrt{6}}{2}$

S04J11. **42153**

S04J12. **-7**

JUNIOR DIVISION	CONTEST NUMBER THREE	SPRING 2004
PART I: 10 minutes	**NYCIML Contest Three**	**Spring 2004**

S04J13. The 4-digit number represented by $abcd$ is multiplied by 9 and the result is $dcba$. Compute (a,b,c,d).

S04J14. A quiz consists of 10 true/false questions. If exactly seven of the answers are true, how many possible answer keys are there?

PART II: 10 minutes **NYCIML Contest Three** **Spring 2004**

S04J15. Compute the sum of the even positive factors of 2004.

S04J16. The expression $(x+1)(x+2)(x+3)(x+4)+1$ can be written as the square of trinomial ax^2+bx+c. Determine the ordered triple (a,b,c).

PART III: 10 minutes **NYCIML Contest Three** **Spring 2004**

S04J17. Compute the smallest natural number that is neither a prime number, nor a factor of 15!

S04J18. Compute, to the nearest second, the first time after 2:00 when the minute and hour hands of a clock make a 45 degree angle.

ANSWERS:

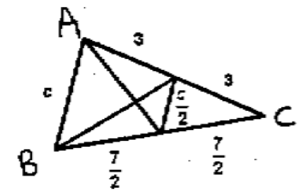
S04J13. **(1, 0, 8, 9)**
S04J14. **120**
S04J15. **4032**
S04J16. **(1,5,5)**
S04J17. **34**
S04J18. **2:02:44**

JUNIOR DIVISION **CONTEST NUMBER ONE** **SPRING 2004**

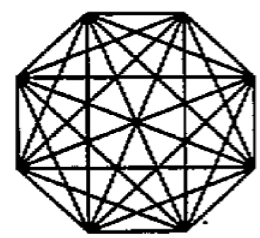
SOLUTIONS

S04J1. A side of the second field is three times a side of the first, so the area of the second field is nine times the area of the first. It would take 5 men 4 x 9 = 36 hours to plow this second field, so it will take 10 men 18 hours.

S04J2. Draw the midline of length $\frac{c}{2}$ between the two midpoints. The trapezoid formed has perpendicular diagonals, so $3^2+\left(\frac{7}{2}\right)^2=c^2+\left(\frac{c}{2}\right)^2$, so $c=\sqrt{17}$.



S04J3. Any four vertices of the octagon determine one intersection of diagonals inside. In the center we are counting what would be normally 6 intersections as just 1. In 8 other cases (intersections closest to the center) we are losing 2 intersections each. There are $\binom{8}{4}-5-16=\mathbf{49}$ intersections.

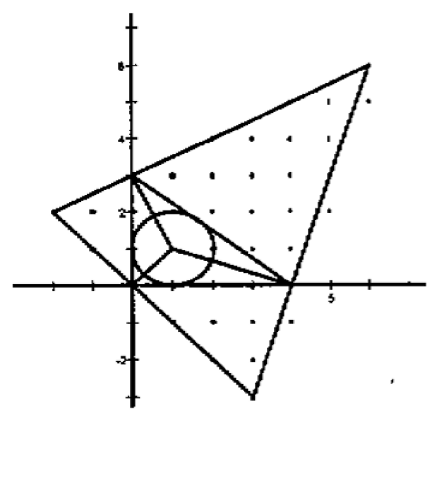


S04J4. Working alone, Willie would build a spaceship in $\frac{33}{.6}=55$ days, and working alone, Kate would build a spaceship in $\frac{33}{.75}=44$ days. If they work together, they can finish building a spaceship in:

$$\frac{1}{\frac{1}{33}+\frac{1}{44}+\frac{1}{55}}=\frac{\mathbf{660}}{\mathbf{47}}=\mathbf{14}\frac{\mathbf{2}}{\mathbf{47}}\text{ days.}$$

S04J5. To convert 555 dollars from base 8 to base 10, we get the following: $5\cdot 8^2+5\cdot 8+5=365$ Similarly, do the same for 455 in base 9, getting: $4\cdot 9^2+5\cdot 9+5=374$. The difference between the two amounts is **9** dollars.

S04J6. Label $A(0,3)$, $B(0,0)$, and $C(4,0)$. The inradius is 1, so the incenter, which lies on all the bisectors, is $I(1,1)$. Now it's simple to find the slopes of the bisectors, obtain the slopes of their perpendiculars, and find the intersections of the perpendiculars (or just sketch it, see diagram.) The perimeter of the big triangle is $3\sqrt{10}+4\sqrt{5}+5\sqrt{2}$.
(10,5,2)



SOLUTIONS

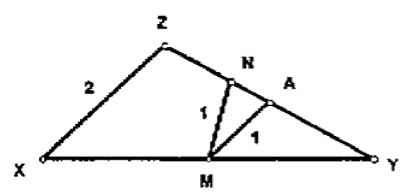
S04J7. Let us assume that each glass holds a liter. Therefore, the first glass contains 1/3 of a liter of milk, and the second glass contains ¼ of a liter of milk. The two glasses together then contain 7/12 liters of milk. Since both glasses will be completely filled with water, resulting in 2 liters of liquid total; there must be 17/12 liters of water. We note that the ratio will not change whether half the mixture is poured back into one of the glasses, or in fact, any amount is poured back. Therefore, the part of mixture in the glass that is milk is simply the ratio of milk in the mixture, 7/12 divided by 2 liters, or 7 /24.

S04J8. Let x = the number. The number x must be one less than a common multiple of 5, 6, and 7. So x = 210k – 1 for some positive integer k. Using k = 10 gives x = **2099** as the answer.

S04J9. Let pencils cost x each and pens cost y each, then $x + 3y = 15$, and $\frac{24}{y} - 3 = \frac{30}{x}$,

so $\frac{24}{y} - 3 = \frac{30}{15-3y}$, or $3y^2 - 49y + 120 = 0$, yielding $y = -\frac{40}{3}$ (no good) and $y = \mathbf{3}$.

S04J10. Construct A on $\overline{YZ}$ so that $\overline{XZ} \parallel \overline{MA}$. Note that ΔNMA is isosceles, so $AM = MN = 1$. Since $\overline{MA}$ is a midline, $XZ = 2$, which leads to $YZ = 2\sqrt{2}$, and $NY = MY = \frac{XY}{2} = \frac{\sqrt{2}+\sqrt{6}}{2}$, so $NZ = \mathbf{\frac{3\sqrt{2}-\sqrt{6}}{2}}$.



S04J11. The first 24 numbers start with 1, the second 24 start with 2, and so on. The 80th number is the 8th number that starts with 4. The first six have 1 as the second digit, the 80th number is the second smallest of those that start with 42, making it **42153**.

S04J12

If x > 1, we have (x – 1) – 2(x + 3) + x + 7 = 0 which becomes 0 = 0 and is true for all x > 1.

If x ≤ 1 and x ≥ -3, we have 1 – x – 2(x + 3) + x + 7 = 0, for which x = 1 is the only solution.

If x < -3, we have 1 – x + 2(x + 3) + x + 7 = 0, or x = -7

Therefore the smallest x is –7.

SOLUTIONS

S04J13. Since the 4-digit number $dcba$ is 9 times another 4-digit number, a would have to be 1 and d would have to be 9. The problem then simplifies to $1bc9$ is multiplied by 9 and the result is $9cb1$.

$$9\cdot(1000+b(100)+c(10)+9)=9000+c(100)+b(10)+1$$
$$9000+900b+90c+81=9000+100c+10b+1$$
$$890b+80=10c$$
$$89b+8=c$$

Since b and c are both digits, b is 0, and c equals 8. Therefore, the solution is **(1, 0, 8, 9)**.

S04J14. We must choose 7 of the 10 questions. We can do that in
$\binom{10}{7}=\binom{10}{3}=\frac{10(9)(8)}{3(2)(1)}=\mathbf{120}$ ways.

S04J15. The set of even positive factors of 2004 has a one to one correspondence with the set of positive factors of 1002. The sought sum will be twice that sum: $2\sigma(1002) = 2(2+1)(3+1)(167+1) = \mathbf{4032}$.

S04J16. $(x+1)(x+2)(x+3)(x+4)+1$ is a perfect square. (We can try a few cases such as x = -1, 0, 1, 2 to guess this fact.) Substitute $y=x+3$ and the expression simplifies to:

$$(y-2)(y-1)(y)(y+1)+1=(y+1)(y-1)(y)(y-2)+1$$
$$=(y^2-1)(y^2-2y)+1=y^4-2y^3-y^2+2y+1=(y^2-y-1)^2$$

Now substitute $y=x+3$ back into the equation above and we get:

$$(y^2-y-1)^2=((x+3)^2-(x+3)-1)^2=(x^2+6x+9-x-3-1)^2=(x^2+5x+5)^2$$

Therefore the ordered triple is (1, 5, 5).

S04J17. The number is (2)(17) = **34**.

S04J18. At 2:00, the minute hand is $\frac{1}{6}$ (of a complete circle) behind. At the first possible point afterwards where the two hands make a 45 degree angle, the minute hand must be $\frac{1}{8}$ of a complete circle behind the hour hand. It must catch up $\frac{1}{24}$. Since the speed of the minute hand is 12 times that of the hour hand, the desired time must be $\frac{1}{24}\cdot\frac{12}{11}=\frac{1}{22}$ of a complete circle. $\frac{60}{22}=2+\frac{8}{11}=2+\frac{480}{60\cdot 11}=2+\frac{43}{60}+\frac{7}{660}$. The closest such time is **2:02:44**.