New York City
Interscholastic
Mathematics
League

Sophomore-Freshman Division

CONTEST NUMBER 1

PART I *FALL, 2003* *CONTEST 1* *TIME: 10 MINUTES*

F03SF1 What percent of $2B$ is $7A$, if A and B are positive numbers? (Leave your answer, in simplest form, in terms of A and B.)

F03SF2 If all the integers from 1 to 3000 are printed, compute the number of 0's that will appear.

PART II *FALL, 2003* *CONTEST 1* *TIME: 10 MINUTES*

F03SF3 The lengths of the sides of a triangle are $b + 1$, $7 - b$, and $4b - 2$. Compute all the values of b for which the triangle is isosceles.

F03SF4 Compute the number of positive integral factors of $3^4 4^5 5^6$.

PART III *FALL, 2003* *CONTEST 1* *TIME: 10 MINUTES*

F03SF5 Average Joe averaged a score of 86 on his 5 math tests. If he averaged a 92 on his first 2 tests, compute the lowest possible average of his final 2 tests. (All tests are scored out of 100 points)

F03SF6 There are 120 five digit positive integers whose digits are the numbers 1, 2, 3, 4, and 5 (no repetition of digits). If all these positive integers are put in increasing order, compute the 26th positive integer.

ANSWERS:

F03SF1	$\frac{350A}{B}\%$ or $\frac{350A}{B}$
F03SF2	**792**
F03SF3	$\frac{9}{5}$
F03SF4	**385**
F03SF5	**73**
F03SF6	**21354**

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CONTEST NUMBER 2

PART I	*FALL, 2003*	*CONTEST 2*	*TIME: 10 MINUTES*

F03SF7 Compute the smallest positive integer that has 12 positive integral factors.

F03SF8 Given the coordinates of $A(3,2)$, $B(4,4)$, and $C(2,3)$, compute the area of parallelogram $ABCD$.

PART II	*FALL, 2003*	*CONTEST 2*	*TIME: 10 MINUTES*

F03SF9 Compute the units digit of 2003^{2003}.

F03SF10 A palindrome is a number that will remain unchanged when its digits are reversed. For example, 3, 34243, 959, 1771, 2002 are all palindromes. Compute the number of positive integral palindromes less than 10000 such that the ten's digit is larger than the unit's digit.

PART III	*FALL, 2003*	*CONTEST 2*	*TIME: 10 MINUTES*

F03SF11 Find the smallest positive even integer that is both a multiple of 15 and of 21.

F03SF12 Alice and Bob played a game by tossing two fair six-sided dice whose faces are painted either red or blue. Alice wins when the two top faces are the same color. Bob wins when the colors are different. Alice has a 50% chance of winning and the first die has 5 red faces and 1 blue face. How many red faces are there on the second dice?

ANSWERS:

F03SF7	**60**
F03SF8	**3**
F03SF9	**7**
F03SF10	**72**
F03SF11	**210**
F03SF12	**3**

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CONTEST NUMBER 3

PART I ***FALL, 2003*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F03SF13 Compute the sum of the coefficients in the expansion of $(x+2y-z)^3$.

F03SF14 The graph of the line $y = 3x + 7$ is rotated 270 degrees counter-clockwise around the origin. Express the equation of the new line in the form: $y = mx + b$.

PART II ***FALL, 2003*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F03SF15 $23x0255y7283$ is divisible by both 9 and 11. Compute the ordered pair of the digits (x,y).

F03SF16 An ant starts at the origin of the Cartesian plane and moves one unit at a time either left, right, up, or down. Compute the number of ways for the ant to reach (4,7) in exactly 11 moves.

PART III ***FALL, 2003*** ***CONTEST 3*** ***TIME: 10 MINUTES***

F03SF17 Two fair six-sided dice are tossed and their sum is less than 10. Compute the probability that their sum is less than 5.

F03SF18 When $2x - 5$ is divided by $x + 7$, the remainder is 10. Compute all possible values of the positive integer x.

ANSWERS:		
	F03SF13	**8**
	F03SF14	$y = -\frac{x}{3} + \frac{7}{3}$
	F03SF15	**(4,4)**
	F03SF16	**330**
	F03SF17	**1/5**
	F03SF18	**22**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
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CONTEST NUMBER 1

SOLUTIONS

F03SF1

$$x\%\cdot 2B = 7A$$
$$x\% = \frac{7A}{2B}$$
$$x = \frac{350A}{B}$$

F03SF2 Imagine every one-digit-number $\underline{a}$ is written as a three-digit number $\underline{00a}$, and every two-digit number $\underline{bc}$ be written $\underline{0bc}$. Then exactly every digit will appear in the one's place, in the ten's place, and in the hundred's place one tenth of the time. This means $3\left(\frac{3000}{10}\right) = 900$ zeroes. However, we have counted the extra zeros we added at the beginning of numbers, so we must subtract $2\cdot 9$ zeroes for the one-digit numbers and $1\cdot 90$ zeroes for the two-digit numbers. This leaves us with $900 - 90 - 18 =$ **792** zeroes.

F03SF3 Case 1: $b + 1 = 7 - b$ implies that $b = 3$, giving a 4-4-10 triangle
This is a violation of the triangle inequality.

Case 2: $b + 1 = 4b - 2$ implies that $b = 1$, giving a 2-2-6 triangle.
This is also a violation of the triangle inequality.
Case 3: $7 - b = 4b - 2$ implies that $b = 9/5$ giving a $\frac{26}{5}$-$\frac{26}{5}$-$\frac{14}{5}$ triangle.
This case yields a valid triangle, therefore B has only one value: $\frac{9}{5}$.

F03SF4 The number of factors of a number $n = p^a\cdot q^b\cdot r^c$ for primes p, q, and r is given by the formula $(a+1)(b+1)(c+1)$. In this case, we have $5\cdot 11\cdot 7 =$ **385** factors.

F03SF5 Assume that Joe scored 100 (the highest possible score) on the third test. Then in order to achieve an average of 86, or a combined score of 430 on his five exams, he must have a combined score of 146 on his final two tests, leading to an average of **73** on the final tests.

F03SF6 The smallest 24 numbers all begin with a 1. The 25$^{\text{th}}$ number is 21345. The 26$^{\text{th}}$ smallest number is **21354**.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

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CONTEST NUMBER 2

SOLUTIONS

F03SF7 Consider a number in the form $n = p^a \cdot q^b \cdot r^c$, p, q, r are prime numbers. Then the number of factors of $n = (a+1)\,(b+1)\,(c+1) = 12 = 3\cdot 2\cdot 2$. To minimize the number we pick 2, 3, and 5 for p, q, and r. Therefore, $n = \mathbf{60}$.

F03SF8 ***Solution 1***: ABC happens to be a isosceles triangle with a base of $\sqrt{2}$ and legs of $\sqrt{5}$. The height is therefore $3/\sqrt{2}$ by the Pythagorean Theorem, and its area is $3/2$. The area of the parallelogram is twice that of the triangle, so it's **3**.

Solution 2: The midpoint of AC is $\left(5/2, 5/2\right)$. Since the diagonals of a parallelogram bisect each other the line through $\left(5/2, 5/2\right)$ and (4,4) goes through D and $\left(5/2, 5/2\right)$ is the midpoint of BD. Therefore $D = (1,1)$. Also, $AB = BC = \sqrt{5}$, so $ABCD$ is rhombus. The area of a rhombus is half the product of the diagonals. $AC = \sqrt{2}$, $BD = 3\sqrt{2}$, so the area of $ABCD$ is $\frac{1}{2}\cdot\sqrt{2}\cdot 3\sqrt{2} = \mathbf{3}$.

F03SF9 Only the 3 of 2003 determines the units digit of the answer. The units digit of the powers of 3 are in cycles of 4 (3, 9, 7, 1). Since the exponent 2003 is 3 more than a multiple of 4, the units digit of the answer is **7**.

F03SF10 The condition that the ten's digit is larger than the unit's digit dictates that both 1-digit and 2-digit palindromes are excluded. For both 3 and 4-digit palindromes, 0 cannot be used anywhere also owing to that condition. For either the 3 or 4-digit palindromes, exactly 2 different digits will be used and arranged precisely one way. The number of ways of picking 2 digits without replacement out of 9 is ${}_9C_2 = 36$. Twice that number is **72**.

F03SF11 $\text{LCM}(15, 21, 2) = 2\cdot 3\cdot 5\cdot 7 = \mathbf{210}$.

F03SF12 Let the probability of the second die turning up red be x. Then:

$$\frac{5}{6}(x)+\frac{1}{6}(1-x)=\frac{1}{2}$$

$$5x+1-x=3$$

$$x = 1/2$$

Therefore, there must be exactly $\frac{1}{2}\cdot 6 = \mathbf{3}$ red faces on the second die.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

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CONTEST NUMBER 3

SOLUTIONS

F03SF13 To obtain the sum of the coefficients, let $x = y = z = 1$:
$(1 + 2 - 1)^3 = \mathbf{8}$.

F03SF14 A rotation of 270 degrees counter-clockwise is a rotation of 90 degrees clockwise about the origin, (a,b) becomes (b,-a). The equation under such a transformation will become $x = 3(-y) + 7$, or equivalently, $y = -\frac{x}{3} + \frac{7}{3}$.

F03SF15 Using a divisibility rule for 11 we obtain that:
$(3+0+5+y+2+3) - (2+x+2+5+7+8) = (13+y) - (24+x) = y - x - 11$ is divisible by 11
$y - x = 11m$ for some integer m. The only such m that will fit here is 0, so $y = x$.
Using a divisibility rule for 9 we obtain that:
$3+0+5+y+2+3+2+x+2+5+7+8 = 9n$ or $37 + y + x = 36 + 1 + x + y = 9n$
$y + x + 1 = 9k$ for some integer k. The only such k that will fit here is 1 or 2.
However, $y = x$ so $y = x = 4$.
The solution is $(x,y) = \mathbf{(4,4)}$.

F03SF16 If the ant is to travel to (4,7) in exactly 11 moves, then every move must either be a move up or a move to the right. The number of ways one can choose a path is ${}_{11}C_4$, or **330**.

F03SF17 Since their sum must be less than 10, 6 out of the 36 possibilities have been thrown out. The probability that their sum is less than 5 is 6/30 = **1/5**.

F03SF18 Solution 1: $\frac{2x-5}{x+7} = 2 - \frac{19}{x+7}$. However the remainder is 10, not –19. We manipulate to get $\frac{2x-5}{x+7} = 2 - \frac{29}{x+7} + \frac{10}{x+7}$. Therefore $\frac{29}{x+7}$ must be an integer. Therefore x + 7 must divide 29. The expression x +7 must equal –1, 1, -29, or 29. x = **22** is the only positive x.

Solution 2: The remainder when $2x - 5$ is divided by $x + 7$ is the same remainder when $2x - 5 - 2(x+7) = -19$ is divided by $x + 7$. Therefore, $x + 7$ is a factor of $|-19 - 9| = 29$. Every value of $x + 7$ except 29 will make x negative, so $x = 29 - 7 = \mathbf{22}$.