New York City Interscholastic Mathematics League

SENIOR B DIVISION	CONTEST NUMBER ONE	FALL 2003

PART I: 10 minutes **NYCIML Contest One** **Fall 2003**

F03B01. Compute $1 - 2 + 3 - 4 + 5 - 6 + \ldots - 2002 + 2003$.

F03B02. Compute the sum of the digits of the first 100 positive odd integers.

PART II: 10 minutes **NYCIML Contest One** **Fall 2003**

F03B03. Compute the product of $(10101)_2$ and $(101010)_2$ and express the answer in base 2.

F03B04. Compute the number of 3 digit positive integers b such that $\log_8 b$ is a rational number.

PART III: 10 minutes **NYCIML Contest One** **Fall 2003**

F03B05. Compute the number of different line segments that have their endpoints on two distinct vertices of a cube.

F03B06. David and Richard each roll a pair of fair dice. Compute the probability that the sum of the numbers shown on David's dice is equal to the sum of the numbers shown on Richard's dice.

ANSWERS

1. 1002
2. 1000
3. 1101110010_2
4. 3
5. 28
6. 73/648

New York City Interscholastic Mathematics League

SENIOR B DIVISION | **CONTEST NUMBER TWO** | **FALL 2003**

PART I: 10 minutes | **NYCIML Contest Two** | **Fall 2003**

F03B07. Compute all real values of x: $\sqrt{3+\sqrt{x}}=5$.

F03B08. Compute the area of the region bounded by the graphs of $y=x$, $y=-x$, and $y=\sqrt{25-x^2}$.

PART II: 10 minutes | **NYCIML Contest Two** | **Fall 2003**

F03B09. If $\frac{1}{x}$ is the average of $\frac{1}{10}$ and $\frac{1}{12}$, compute x.

F03B10. Compute the area of a right triangle with a hypotenuse of length 5 and the sum of the lengths of the legs is $\sqrt{35}$.

PART III: 10 minutes | **NYCIML Contest Two** | **Fall 2003**

F03B11. Compute $(\tan 5°)(\tan 10°)(\tan 15°)...(\tan 80°)(\tan 85°)$.

F03B12. Point P is 7 units from the center of a circle with radius 10. Compute the number of chords with integral lengths that pass through P.

<u>ANSWERS</u>

7. 484
8. $\frac{25\pi}{4}$
9. 120/11
10. 5/2
11. 1
12. 11

New York City Interscholastic Mathematics League

SENIOR B DIVISION | **CONTEST NUMBER THREE** | **FALL 2003**

PART I: 10 minutes | **NYCIML Contest Three** | **Fall 2003**

F03B13. If $\sqrt{x\sqrt{x\sqrt{x}}} = x^a$, compute a.

F03B14. If $m > 0$ and the line through $(5,m)$ and $(m,4)$ has a slope of m, compute m.

PART II: 10 minutes | **NYCIML Contest Three** | **Fall 2003**

F03B15. Compute the sum of the odd digits of the first 100 positive integers.

F03B16. A line connecting the midpoints of the two legs of a trapezoid divides the trapezoid into two smaller trapezoids whose areas are in the ratio 1:2. If the smaller base of the original trapezoid has length 8, compute the length of the larger base of the original trapezoid.

PART III: 10 minutes | **NYCIML Contest Three** | **Fall 2003**

F03B17. Compute the smallest positive integer x such that $180x$ is a perfect cube.

F03B18. D is a point on side $\overline{BC}$ of triangle ABC such that $\overline{AD}$ bisects angle A. If angle B measures 30°, angle C measures 45°, and $\overline{CD}$ has length 5, compute the length of $\overline{BD}$.

<u>**ANSWERS**</u>

13. 7/8
14. $2+2\sqrt{2}$
15. 501
16. 40
17. 150
18. $5\sqrt{2}$

New York City
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SENIOR B DIVISION **CONTEST NUMBER FOUR** **FALL 2003**

PART I: 10 minutes **NYCIML Contest Four** **Fall 2003**

F03B19. A group of 20 numbers has an average of 20. A distinct group of 30 numbers has an average of 30. Yet another distinct group of 40 numbers has an average of 40. If all three groups of numbers are combined in a large group, compute the average of this large group.

F03B20. Compute the number of two digit positive integers which are not divisible by 10 and have the property that the product of their digits is a perfect square.

PART II: 10 minutes **NYCIML Contest Four** **Fall 2003**

F03B21. Compute $(\sin 15° + \sin 75°)^2$.

F03B22. In triangle ABC, $AB = 6$, $AC = 7$, $BC = 9$. D is on $\overline{BC}$ so that $AD = 6$. Compute BD.

PART III: 10 minutes **NYCIML Contest Four** **Fall 2003**

F03B23. Compute the number of values of N, between -100 and 100 inclusive, such that the equation $x^2 + 6x + N = 0$ has integral roots.

F03B24. The sides of a triangle have length 8, 10, and x. Compute the number of integral values of x for which the triangle is obtuse.

ANSWERS

19. 290/9
20. 17
21. 3/2
22. 68/9
23. 11
24. 8

New York City
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Mathematics
League

SENIOR B DIVISION **CONTEST NUMBER FIVE** **FALL 2003**

PART I: 10 minutes **NYCIML Contest Five** **Fall 2003**

F03B25. Ten married couples are in a room. If everyone shakes hands with everyone else, except his/her spouse, how many handshakes have taken place?

F03B26. (3,-2) and (-7,-2) are endpoints of a diameter of a circle. If $(x,1)$ is a point on the circle, compute all possible values of x.

PART II: 10 minutes **NYCIML Contest Five** **Fall 2003**

F03B27. Compute the number of different values of N such that a regular N-sided polygon will have interior angles with an integral number of degrees.

F03B28. In equilateral triangle ABC, D is on $\overline{AB}$, E is on $\overline{BC}$, and F is on $\overline{AC}$. DEF is an equilateral triangle and $\overline{ED}$ is perpendicular to $\overline{AB}$. If the area of triangle ABC is k, compute the area of triangle DEF in terms of k.

PART III: 10 minutes **NYCIML Contest Five** **Fall 2003**

F03B29. In right triangle ABC, $\sin A + \sin B + \sin C = 2.3$. Compute $\cos A + \cos B + \cos C$.

F03B30. In an increasing series of positive integers, each term after the second is the sum of the two which immediately preceded it. If the seventh term is 71, compute the eighth term.

ANSWERS

25. 180
26. 2, -6
27. 22
28. k/3
29. 1.3
30. 115

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER ONE** **Fall 2003**

SOLUTIONS

F03B1
Take the sum in pairs. The first 1001 pairs have a sum of -1001. Adding 2003, we get **1002.**

F03B2
The first number is 1 and the last number is 199. The average value of the unit's digit is 5, the average value of the ten's digit is 4.5, and the average value of the hundred's digit is 0.5. So the sum of the digits is 100*(5+4.5+0.5) = **1000.**

F03B3
SOLN1: The first number is 111 base 4, the second one is twice that. In base 4, the answer is twice 12321_4. Therefore, in binary, the number is $\mathbf{1101110010_2}$.

SOLN 2: Change both numbers to base ten, multiply them, and change back:
$10101_2 * 101010_2 = 21 * 42 = 882 = \mathbf{1101110010_2}$.

SOLN3: Multiply in binary.

F03B4
b must be a power of 2. The only solutions are **128, 256,** and **512.** The answer is **3.**

F03B5
SOLN1: ${}_8C_2 = \mathbf{28}$
SOLN2: 12 edges + 12 facial diagonals + 4 internal diagonals = **28**

F03B6
$$\frac{2\left(1^2+2^2+3^2+4^2+5^2\right)+6^2}{36^2}=\frac{146}{1296}=\mathbf{\frac{73}{648}}$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER TWO** **Fall 2003**

SOLUTIONS

F03B7

$3+\sqrt{x}=25$, so $x = \mathbf{484}$.

F03B8

$x^2+y^2=25$ is a circle of radius 5 centered at the origin. $y=\sqrt{25-x^2}$ is a semi-circle while the lines $y=x$ and $y=-x$ define a sector of that circle within the semicircle that is exactly $\frac{1}{4}$ of the circle. The area is therefore $\mathbf{\frac{25\pi}{4}}$.

F03B9

$$\frac{1}{x}=\frac{\frac{1}{10}+\frac{1}{12}}{2}=\frac{12+10}{240}=\frac{11}{120}$$

$$x=\mathbf{\frac{120}{11}}$$

F03B10

Let a and b be the lengths of the legs: $a+b=\sqrt{35}$, $\sqrt{a^2+b^2}=5$

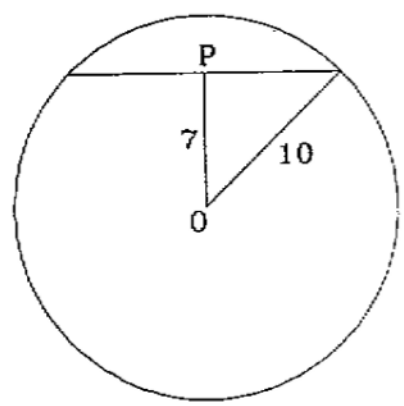
$$\frac{ab}{2}=\frac{a^2+2ab+b^2-\left(a^2+b^2\right)}{4}=\frac{(a+b)^2-\left(\sqrt{a^2+b^2}\right)^2}{4}=\frac{35-25}{4}=\mathbf{\frac{5}{2}}$$

F03B11

$1=(\tan 5°)(\tan 85°)=(\tan 10°)(\tan 80°)=\ldots=(\tan 40°)(\tan 50°)=(\tan 45°)$

The product is therefore also **1**.

F03B12



The longest chord with integral length is 20, the length of the diameter through P. The shortest is $2\sqrt{51}$, which is $\sqrt{204}$, somewhere between 14 and 15. So the chord lengths can be anywhere from 15 to 20, with 2 chords for each length except 20. The number of chords is $2\cdot 5+1=\mathbf{11}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

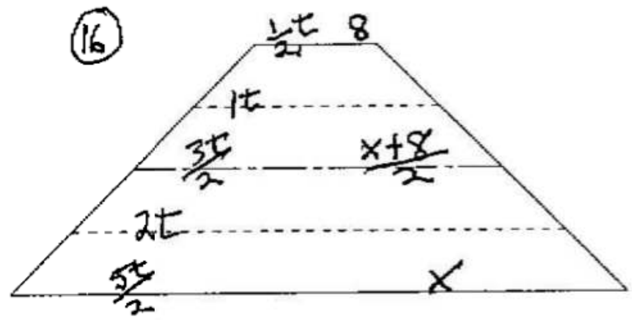
SENIOR B DIVISION **CONTEST NUMBER THREE** **Fall 2003**

SOLUTIONS

F03B13

$\sqrt{x\sqrt{x\sqrt{x}}} = \sqrt{x\sqrt{x^{3/2}}} = \sqrt{x^{7/4}} = x^{\frac{7}{8}}$,

Therefore **a = 7/8**



F03B14

$\frac{m-4}{5-m} = m$, so $m^2 - 4m - 4 = 0$

$m = \frac{4 \pm \sqrt{16-4(1)(-4)}}{2} = 2 \pm 2\sqrt{2}$.

Since m is positive, $m = \mathbf{2+2\sqrt{2}}$

F03B15 The average values of both the unit's digit and the ten's digit (even values are evaluated as 0) are both 2.5. The only nonzero hundred's digit occurs at 100. So the sum of the digits is $100 \cdot (2.5+2.5)+1 =$ **501**.

F03B16

SOLN1: The ratio of the area of the top (smaller) trapezoid to that of the bottom (larger) trapezoid is the ratio of their respective medians, since their heights are equal. Notice further that the 5 lines in the picture above are in arithmetic progression. The smaller base is 8 so $\frac{1}{2}t = 8,\ t = 16$. The larger base is therefore $\frac{5}{2}t = \frac{5}{2}(16) = 40$.

SOLN2: Let the larger base be x and the height of each part of the trapezoid be h:

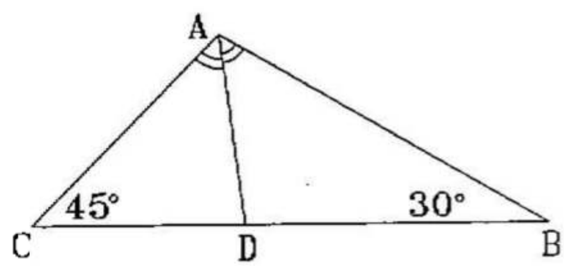
$$\frac{1}{2}h\left(x + \frac{x+8}{2}\right) = 2 \cdot \frac{1}{2}h\left(8 + \frac{x+8}{2}\right)$$

$x + \frac{x+8}{2} = 16 + 8 + x$, $x =$ **40**

F03B17

The exponents of all prime numbers in its factorization must be multiples of 3:

$180 = 2^2 3^2 5^1$, so $x = 2^1 3^1 5^2 =$ **150**.



F03B18

By the angle bisector theorem: $\frac{AB}{AC} = \frac{BD}{CD}$

By the law of sines: $\frac{AB}{\sin C} = \frac{AC}{\sin B}$

Together: $BD = \frac{CD \cdot AB}{AC} = \frac{CD \cdot \sin C}{\sin B} = \frac{5 \cdot \sin 45°}{\sin 30°} = \frac{5 \cdot \sqrt{2}/2}{1/2} = \mathbf{5\sqrt{2}}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER FOUR** **Fall 2003**

SOLUTIONS

F03B19

$$\frac{20\cdot 20+30\cdot 30+40\cdot 40}{90}=\frac{2900}{90}=\mathbf{\frac{290}{9}}$$

F03B20

All 9 integers whose digits are the same work, along with 14, 41, 19, 91, 49, 94, 28, and 82. Together, there are **17** numbers.

F03B21

$$(\sin 15°+\cos 15°)^2=\sin^2 15°+2\sin 15°\cos 15°+\cos^2 15°$$

$$=1+\sin 30°=1+\frac{1}{2}=\mathbf{\frac{3}{2}}$$

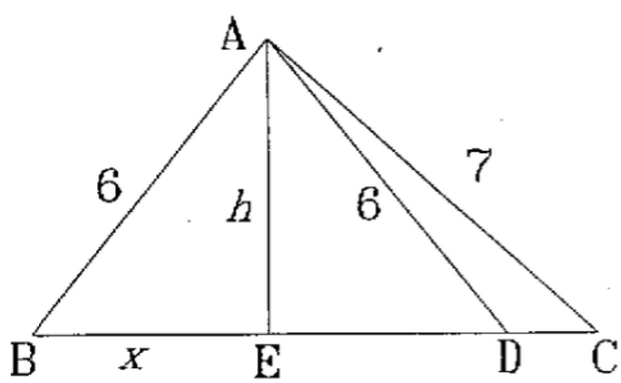
F03B22

Let the foot of the height of the triangle from vertex A be E. Let $BE=x$ and $AE=h$.

$h^2=36-x^2=49-(9-x)^2$

$68=18x,\ x=34/9$

$BD=2x=\mathbf{68/9}$.



F03B23

SOLN1: The sum of the two roots is -6. Let one such root be x, then the other is $-6-x$. Let $f(x)=x(-6-x)=-x^2-6x$. Then the integral values of x such that $f(x)$ would lie in the interval (-100,100) range from $x=-13$ to $x=7$. These 21 integral values of x are made up of 10 pairs of equal $f(x)$ value and one singularity, so there are **11** different values of $f(x)$.

SOLN2: N can be 0.

If N is positive, it can be 5, 8, or 9

If N is negative, it can be -7, -16, -27, -40, -55, -72, -91

There are **11** values.

F03B24

To be a triangle: $2<x<18$

To be obtuse: $x^2>100+64$ or $100>x^2+64$

The obtuse criterion could also be: $x>12$ or $x<6$

This happens for **8** values of $x=3, 4, 5, 13, 14, 15, 16, 17$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER FIVE** **Fall 2003**

SOLUTIONS

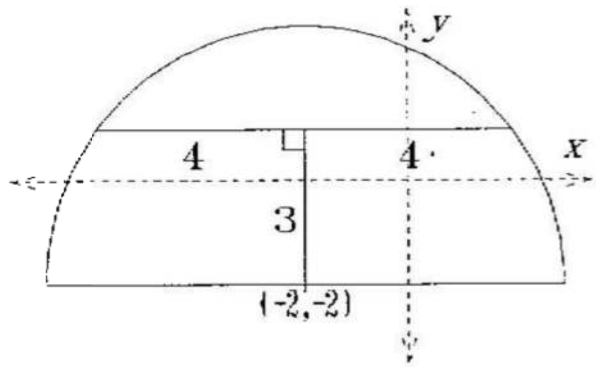
F03B25

SOLN 1: If everyone shakes hands with everyone else, then there are ${}_{20}C_2 = 190$. Since couples do not shake hands between themselves, the total number of handshakes is 190-10 = **180**.

SOLN 2: There are 4 handshakes between any two couples. Call this a "couple-shake". There are 10 couples and therefore ${}_{10}C_2 = 45$ couple-shakes. This amounts to 4*45 = **180** handshakes in total.

F03B26

The center of the circle is (-2,-2). x is $\pm\sqrt{25-9} = \pm 4$ away from -2, so the possible values of x are **2, -6**.



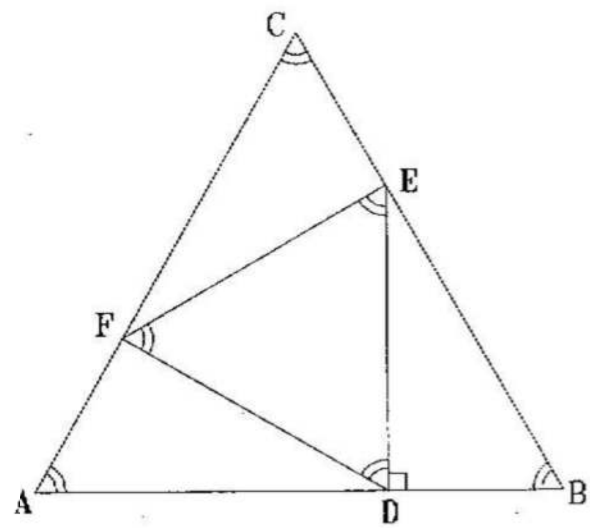
F03B27

The internal angle is integral precisely when the external angle is integral (since they sum up to 180°), and the external angle is equal to 360° divided by the number of sides. In other words, the internal angle is integral when the number of sides is a factor of 360 larger than 2. The number of factors of $360 = 2^3 3^2 5^1$ is $4 \cdot 3 \cdot 2 = 24$. Subtracting the two degenerate cases, the answer is $24 - 2 = \mathbf{22}$.

F03B28

$ED{:}BD = \sqrt{3}$, and $AB{:}BD = (BD+EB){:}BD = (1+2){:}1 = 3$, so $AB{:}ED = \sqrt{3}$

Therefore, the ratio of the areas is 3 and triangle DEF has area $\frac{k}{3}$.



F03B29

Suppose (without loss of generality that) C is the right angle, then:

$\sin A + \sin B = 2.3 - \sin C = 2.3 - 1 = 1.3 = \cos A + \cos B$

$\cos A + \cos B + \cos C = 1.3 + 0 = \mathbf{1.3}$

F03B30

Let the first number be x and the second number be y with $0 < x < y$, then the sequence is:

$x, y, x+y, x+2y, 2x+3y, 3x+5y, 5x+8y, 8x+13y, 13x+21y, 21x+34y, \ldots$

The seventh term: $5x + 8y = 71$ yields a singular solution of $x=3$ and $y=7$.

The eighth term must be $8 \cdot 3 + 13 \cdot 7 = \mathbf{115}$.