New York City
Interscholastic
Mathematics
League

SENIOR A DIVISION | **CONTEST NUMBER ONE** | **FALL 2003**

PART I: 10 minutes | **NYCIML Contest One** | **Fall 2003**

F03S1. Compute all ordered pairs (a,b) so that the four-digit number $a97b$ is divisible by 45.

F03S2. Compute: $\sin^3\alpha - \cos^3\alpha$, if $\sin\alpha - \cos\alpha = 0.6$

PART II: 10 minutes | **NYCIML Contest One** | **Fall 2003**

F03S3. Given $\log_{11} 2 = a$, $\log_{11} 3 = b$, and $\log_{11} 10 = c$, express $\log_{10} 36$ in simplest form in terms of a, b, and c.

F03S4. In triangle ABC, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are medians, $AD = CF = 12$, and the area of triangle ABC is $12\sqrt{39}$. Compute the maximum value of BE.

PART III: 10 minutes | **NYCIML Contest One** | **Fall 2003**

F03S5. Point O is in the interior of regular pentagon $ABCDE$ so that triangle OAB is equilateral. Compute the measure of $\angle OCD$.

F03S6. The angles of a triangle with sides 3, 7, and x form an arithmetic progression. Compute all possible values of x.

ANSWERS:

F03S1. **(2, 0) and (6, 5)**

F03S2. **0.792**

F03S3. $\frac{2a+2b}{c}$

F03S4. $6\sqrt{13}$

F03S5. **42**

F03S6. $\sqrt{37}$, **8**

New York City Interscholastic Mathematics League

SENIOR A DIVISION	CONTEST NUMBER TWO	FALL 2003
PART I: 10 minutes	NYCIML Contest Two	Fall 2003

F03S7. Compute all x: $\left(x^2+5x-2\right)^2-4\left(x^2+5x-2\right)=96$.

F03S8. Compute all ordered pairs (x,y) of positive integers with $x+y=A$, $x-y=B$, $xy=C$, and $x/y=D$, and $A+B+C+D=243$.

PART II: 10 minutes	NYCIML Contest Two	Fall 2003

F03S9. Compute the number of ordered triples (x,y,z) of positive integers that satisfy $x+y+z=10$.

F03S10. Compute: $(\cos 20°)(\cos 40°)(\cos 60°)(\cos 80°)$.

PART III: 10 minutes	NYCIML Contest Two	Fall 2003

F03S11. $0.1+0.02+0.003+0.0004+\ldots+0.000000009+0.0000000010+\ldots$ is equal to the reduced fraction $\frac{a}{b}$. Compute $\frac{a}{b}$.

F03S12. A space station has the shape of a cube with side s. An astronaut outside the station is tethered to the center of one of the faces of the cube by a rope of length s. Express in simplest form, in terms of s, the surface area of the station that the astronaut can reach.

ANSWERS:

F03S7. **-7,-3,-2,2**

F03S8. **(54, 2) and (24, 8)**

F03S9. **36**

F03S10. $\mathbf{\frac{1}{16}}$

F03S11. 10/81

F03S12. $\left(\frac{2\pi}{3}+\sqrt{3}-1\right)s^2$ or equivalent

New York City
Interscholastic
Mathematics
League

SENIOR A DIVISION **CONTEST NUMBER THREE** **FALL 2003**
PART I: 10 minutes **NYCIML Contest Three** **Fall 2003**

F03S13. Triangle ABC is a triangle having sides of 40, 42, and 58. Find the diameter of the circle inscribed in the triangle.

F03S14. $2^{18}+1$ has one factor between 400 and 500. Compute this factor.

PART II: 10 minutes **NYCIML Contest Three** **Fall 2003**

F03S15. A two-digit number is divided by the sum of its digits. What is the largest possible remainder?

F03S16. In triangle ABC, $\overline{AD}$ is an angle-bisector, $AB = 18$, $AD = 12$, and $AC = 24$. Compute BC.

PART III: 10 minutes **NYCIML Contest Three** **Fall 2003**

F03S17. Let a and b be the imaginary cube roots of 1. Compute $a^4 + b^4 + \frac{1}{ab}$.

F03S18. If a, b, and c are the sides of triangle ABC such that a^2, b^2, and c^2 form an arithmetic progression, and $\cot A + \cot C = 3$, compute $\cot B$.

ANSWERS:
F03S13. **24**
F03S14. **481**
F03S15. **15**
F03S16. $\mathbf{14\sqrt{6}}$
F03S17. **0**
F03S18. **1.5**

New York City Interscholastic Mathematics League

SENIOR A DIVISION **CONTEST NUMBER FOUR** **FALL 2003**

PART I: 10 minutes **NYCIML Contest Four** **Fall 2003**

F03S19. Given that $\sqrt[3]{26+15\sqrt{3}} = a + b\sqrt{3}$, where a and b are positive integers, compute (a,b).

F03S20. Express in simplest form:

$$\left(\frac{\sqrt{x+\sqrt{x^2-1}}+\sqrt{x-\sqrt{x^2-1}}}{\sqrt{x+\sqrt{x^2-1}}-\sqrt{x-\sqrt{x^2-1}}}\right)^2, \quad \text{where} \quad x \geq 1.$$

PART II: 10 minutes **NYCIML Contest Four** **Fall 2003**

F03S21. The roots of $x^2 + ax + b = 0$ are a and b, where $a \neq 0$ and $b \neq 0$. Compute (a,b).

F03S22. The edges of a cube have length 1, and A is a vertex of the cube. Compute the length of the segment joining A to the center of a face that does not contain A.

PART III: 10 minutes **NYCIML Contest Four** **Fall 2003**

F03S23. Compute the number of terms in the expansion of $(w+x+y+z)^{12}$.

F03S24. A circle is inscribed in a square. A second circle, inside the square, is externally tangent to the first circle, and is also tangent to two sides of the square. Compute the ratio of the radius of the smaller circle to the radius of the first circle.

ANSWERS:

F03S19. **(2, 1)**

F03S20. $\mathbf{\dfrac{x+1}{x-1}}$

F03S21. **(1,-2)**

F03S22. $\mathbf{\dfrac{\sqrt{6}}{2}}$

F03S23. **455**

F03S24. $\mathbf{3-2\sqrt{2}}$

SENIOR A DIVISION **CONTEST NUMBER FIVE** **FALL 2003**

PART I: 10 minutes **NYCIML Contest Five** **Fall 2003**

F03S25. Compute: $\log\tan 1° + \log\tan 2° + \log\tan 3° + \cdots + \log\tan 88° + \log\tan 89°$.

F03S26. Matthew is at a point A that is 4 miles north of a river that is one mile wide and flows from east to west. He wishes to travel to a point B that is 10 miles south and 5 miles east of point A. He can have a bridge built that crosses the river anywhere he chooses, but the bridge must be north-south. Compute the number of miles in the length of the shortest possible path from A to B that crosses the bridge, assuming the bridge is placed in the place that minimizes the length of that path.

PART II: 10 minutes **NYCIML Contest Five** **Fall 2003**

F03S27. Michael is four times as old as Lisa was when Michael was as old as Lisa is now, and Michael is also twice as old as Lisa was when Michael was six years older than Lisa is now. Compute Michael's age.

F03S28. Compute the radius of the inscribed sphere of a regular tetrahedron whose edges have length 6.

PART III: 10 minutes **NYCIML Contest Five** **Fall 2003**

F03S29. Compute: $\dfrac{\cos 15° + \sin 15°}{\cos 15° - \sin 15°}$.

F03S30. In right triangle ABC with right angle C, points P and Q are on $\overline{AB}$ so that $AP = PQ = QB = \frac{1}{3}AB$, CP=7, and CQ=9. Compute AB.

ANSWERS:

F03S25. **0**

F03S26. $\mathbf{1+\sqrt{106}}$

F03S27. **24**

F03S28. $\mathbf{\frac{\sqrt{6}}{2}}$

F03S29. $\mathbf{\sqrt{3}}$

F03S30. $\mathbf{3\sqrt{26}}$

New York City
Interscholastic
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SENIOR A DIVISION **CONTEST NUMBER ONE** **FALL 2003**

SOLUTIONS

F03S1. To be divisible by 45, the number must be divisible by 9 and 5. Thus b=0 or b=5, and 9 divides a+9+7+b, which implies that $a+b=2$ or $a+b=11$. The ordered pairs that satisfy both conditions are (2, 0) and (6, 5).

F03S2. $\sin\alpha-\cos\alpha=0.6$, so $\sin^2\alpha-2\sin\alpha\cos\alpha+\cos^2\alpha=0.36$. Now since $\sin^2\alpha+\cos^2\alpha=1$, $\sin\alpha\cos\alpha=0.32$.
$\sin^3\alpha-\cos^3\alpha=(\sin\alpha-\cos\alpha)(\sin^2\alpha+\sin\alpha\cos\alpha+\cos^2\alpha)=(0.6)(1.32)=\mathbf{0.792}$

F03S3. Note that $\log_{10}36=\dfrac{\log_{11}36}{\log_{11}10}=\dfrac{2\log_{11}2+2\log_{11}3}{\log_{11}10}=\dfrac{2a+2b}{c}$.

F03S4. Let G be the intersection of the medians, and let GE=x. The medians intersect at a point 2/3 along each median from its vertex point, so CG=8, and BG=2x. Since AD=CF, triangle ABC is isosceles, so $\overline{BE}$ is an altitude. Then $K=(1/2)bh$ implies $12\sqrt{39}=(1/2)\cdot 2\sqrt{8^2-x^2}\cdot 3x$. Square and simplify to obtain $x^4-64x^2+624=0$, which implies $x^2=12$ or $x^2=52$. The maximum value of BE is thus $3\sqrt{52}=6\sqrt{13}$.

F03S5. Since $m\angle OBA=60$ and $m\angle ABC=108$, $m\angle OBC=48$. Also, OB=AB=BC, so $m\angle BOC=m\angle BCO=(1/2)(180-48)=66$. Thus $m\angle OCD=108-66=42$.

F03S6. Since the sum of the sum of the angles is $180°$, the middle one in the arithmetic sequence will have a measure of $60°$, and in fact that is a sufficient condition for the angles to be in an arithmetic sequence. There are three possible positions for the angle, each leading, via the Law of Cosines to one of the three equations:

$3^2=7^2+x^2-2(7)(x)\cos 60°\rightarrow x^2-7x+40=0\rightarrow$ no real solutions

$7^2=3^2+x^2-2(3)(x)\cos 60°\rightarrow x^2-3x-40=0\rightarrow x=8$

$x^2=7^2+3^2-2(7)(3)\cos 60°\rightarrow x^2=37\rightarrow x=\sqrt{37}$

Thus the only solutions are $\mathbf{\sqrt{37}}$, **8.**

F03S7. Let $y = x^2 + 5x - 2$. Then $y^2 - 4y = 96$, so $y = -8$ or $y = 12$. Now solve $x^2 + 5x - 2 = 12$ to obtain $x = -7$ or $x = 2$, and solve $x^2 + 5x - 2 = -8$ to obtain $x = -2$ or $x = -3$. The four solutions are -7, -3, -2 and 2.

F03S8. We have $x+y+x-y+xy+x/y=243$, which yields $x(y+1)^2 = 243y$. Since $(y+1)^2$ and y are relatively prime, $(y+1)^2$ must divide $243=3^5$. Thus $(y+1)^2 = 3^2$ or 3^4, so $y=2$ or $y=8$, and the desired ordered pairs are (54, 2) and (24, 8).

F03S9. Each ordered triple corresponds to a string of 10 x's with two d's inserted as dividers. For example (6,3,1) corresponds to *xxxxxx d xxx d x*. There are 9 spaces in which to insert the two dividers, so there are $\binom{9}{2}=36$ ordered triples.

F03S10. Let $P = (\cos 20°)(\cos 40°)(\cos 60°)(\cos 80°)$ Then

$$8P\sin 20° = 8(\sin 20°)(\cos 20°)(\cos 40°)(\cos 60°)(\cos 80°) =$$

$$= 4(\sin 40°)(\cos 40°)(\cos 60°)(\cos 80°) = 2(\sin 80°)(\cos 60°)(\cos 80°) =$$

$$(\cos 60°)(\sin 160°) = (\cos 60°)(\sin 20°) = \frac{1}{2}\sin 20°$$

$$8P\sin 20° = \frac{1}{2}\sin 20° \rightarrow P = \mathbf{\frac{1}{16}}$$

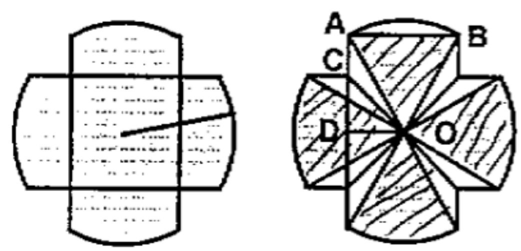
F03S11. $.1 + .02 + .003 + .0004 + \ldots =$

$$= (.1 + .01 + .001 + \ldots) + (.01 + .001 + .0001 + \ldots) + (.001 + .0001 + .00001 + \ldots)$$

$$= (.1 + .01 + .001 + \ldots)(1 + .1 + .01 + \ldots) = 10 (.1 + .01 + .001 + \ldots)^2$$

$$= 10\left(\frac{.1}{1-.1}\right)^2 = 10\left(\frac{1}{9}\right)^2 = \mathbf{\frac{10}{81}}$$

F03S12. Cutting out the reachable area and spreading it out would give us the shape on the right. $AB = AO = BO$, so $m\angle AOB = 60°$, and the sum of the shaded areas in the second picture is $4\left(\frac{1}{6}\right)(\pi s^2) = \frac{2\pi s^2}{3}$. $AO = s$, $DO = \frac{s}{2}$, and $m\angle AOD = 60°$. This makes $AD = \frac{s\sqrt{3}}{2}$, and as



$CD = \frac{s}{2}$, $AC = \frac{s(\sqrt{3}-1)}{2}$, so the sum of the areas of the white triangles is

$8\left(\frac{1}{2}\right)\left(\frac{s(\sqrt{3}-1)}{2}\right)\left(\frac{s}{2}\right) = s^2(\sqrt{3}-1)$ The total area of the figure is $\left(\mathbf{\frac{2\pi}{3} + \sqrt{3} - 1}\right)s^2$.

SENIOR A DIVISION **CONTEST NUMBER THREE** **FALL 2003**

SOLUTIONS

F03S13. Triangle ABC is a right triangle. 40 – r + 42 – r = 58. 2r = 24. Diameter = 24.

F03S14. The square root of the number is just outside of the required range, which suggests a difference of squares approach might yield a result. In fact,

$2^{18}+1=2^{18}+2\left(2^{9}\right)+1-2\left(2^{9}\right)=\left(2^{9}+1\right)^{2}-\left(2^{5}\right)^{2}=$

$=\left(2^{9}+1\right)^{2}-\left(2^{5}\right)^{2}=\left(2^{9}+2^{5}+1\right)\left(2^{9}-2^{5}+1\right)=(545)(481)$. **481** is the sought factor.

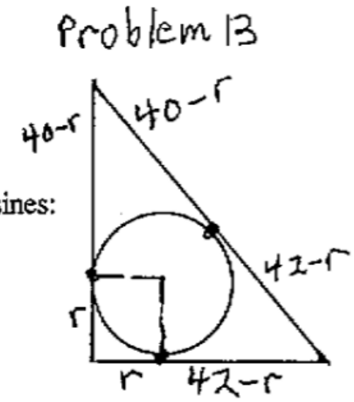
F03S15. If 10t + u is divided by t + u the largest possible remainder is t + u – 1. The largest value of t + u is 18 so that the largest possible remainder is 17. However 99/18 gives a remainder of 9. So 17 is not possible. Also, 98/17 gives a remainder of 13 and 89/17 gives a remainder of 4. So a remainder of 16 is not possible. However, 97/16 gives a remainder of 1, but 79/16 gives a remainder of 15. So the largest possible remainder is 15.

F03S16. Use the Angle-Bisector Theorem to conclude that $BD/DC=AB/AC=3/4$. Let $BD=3x$ and $DC=4x$. Note that cos ADB = -cos ADC, then use the Law of Cosines to conclude $\frac{12^2+9x^2-18^2}{2(12)(3x)}=-\frac{12^2+16x^2-24^2}{2(12)(4x)}$. Solve to find that $x=2\sqrt{6}$. Then $BC=7x=14\sqrt{6}$.

F03S17. Note that a and b must satisfy $x^3=1$ which is equivalent to $(x-1)\left(x^2+x+1\right)=0$.

Thus a and b satisfy $x^2+x+1=0$, so $a+b$ = -1 and ab = 1. Now $a^4+b^4+\frac{1}{ab}$

$=a^3a+b^3b+\frac{1}{ab}=a+b+\frac{1}{ab}=-1+1=0$.



F03S18. The squares of the sides appear in the Law of Cosines:

$\cos A=\frac{b^2+c^2-a^2}{2bc}$, $\cos B=\frac{a^2+c^2-b^2}{2ac}$, $\cos C=\frac{a^2+b^2-c^2}{2ab}$.

Since the question deals with cotangents, and $\cot\theta=\frac{\cos\theta}{\sin\theta}$, we also need sines:

$\sin A=\frac{a}{2R}$, $\sin B=\frac{b}{2R}$, $\sin C=\frac{c}{2R}$, where R is the circumradius.

Dividing, we obtain:

$\cot A=\frac{R\left(b^2+c^2-a^2\right)}{abc}$, $\cot B=\frac{R\left(a^2+c^2-b^2\right)}{abc}$, $\cot C=\frac{R\left(a^2+b^2-c^2\right)}{abc}$

For simplicity, let $k=\frac{abc}{R}\neq 0$, so

$k\cot A=b^2+c^2-a^2=\left(a^2+b^2+c^2\right)-2a^2$,

$k\cot B=a^2+c^2-b^2=\left(a^2+b^2+c^2\right)-2b^2$, and

$k\cot C=a^2+b^2-c^2=\left(a^2+b^2+c^2\right)-2c^2$.

Now since a^2, b^2, and c^2 are an arithmetic progression, so are $2a^2$, $2b^2$, and $2c^2$, and therefore so are the values of the three equations above. Thus $\cot A$, $\cot B$, and $\cot C$ also form an arithmetic progression, and $\cot B=\frac{\cot A+\cot C}{2}=\mathbf{1.5}$

SENIOR A DIVISION **CONTEST NUMBER FOUR SOLUTIONS** **FALL 2003**

F03S19. The given equation implies: $26+15\sqrt{3}=\left(a+b\sqrt{3}\right)^3=a^3+9ab^2+\left(3a^2b+3b^3\right)\sqrt{3}$, so $a\left(a^2+9b^2\right)=26$ and $b\left(a^2+b^2\right)=5$. Thus b divides 5, so b = 1 or b = 5, but b = 5 does not yield an integer for a. When b = 1, then a = 2, so (a,b) = (2, 1).

F03S20. Square the numerator and denominator to obtain:
$\dfrac{x+\sqrt{x^2-1}+x-\sqrt{x^2-1}+2\sqrt{x^2-\left(x^2-1\right)}}{x+\sqrt{x^2-1}+x-\sqrt{x^2-1}-2\sqrt{x^2-\left(x^2-1\right)}}$, which simplifies to $\dfrac{x+1}{x-1}$.

F03S21. The conditions imply that $a + b = -a$ and $ab = b$. Thus (a,b) = (1, -2).

F03S22. Let O be the center of a face not containing A, and let M be the midpoint of one of the sides of that face. Then OMA is a right triangle, $AM^2=1^2+(1/2)^2$,
$OA^2=AM^2+OM^2=1^2+(1/2)^2+(1/2)^2$, and $OA=\sqrt{6}/2$

F03S23. The terms are each of the form $w^a x^b y^c z^d$, where a, b, c, and d are nonnegative integers, and $a+b+c+d$=12. The number of ordered 4-tuples that satisfy $a+b+c+d$=12 is the same as the number of arrangements of 12 x's and 3 d's that serve as dividers, namely,
$\binom{15}{3}=455$.

F03S24. Let $ABCD$ be a square whose center is O, and let the smaller circle have center P and be tangent to $\overline{AB}$ and $\overline{BC}$. Draw radii $\overline{OS}$ and $\overline{PQ}$ to the points of tangency on $\overline{AB}$. Then $\overline{OS}$ and $\overline{PQ}$ are perpendicular to $\overline{AB}$. Let R and r be the radii of the larger and smaller circles, respectively. Then $OB=R\sqrt{2}$, $OP = R+r$, and $PB=r\sqrt{2}$. Thus $R+r+r\sqrt{2}=R\sqrt{2}$, so
$\dfrac{r}{R}=\dfrac{\sqrt{2}-1}{\sqrt{2}+1}=3-2\sqrt{2}$.

New York City
Interscholastic
Mathematics
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SENIOR A DIVISION **CONTEST NUMBER FIVE SOLUTIONS** **FALL 2003**

F03S25. The sum equals:
$\log(\tan 1°\tan 2°\tan 3°\cdots\tan 88°\tan 89°)=\log(\tan 1°\tan 89°\tan 2°\tan 88°\cdots\tan 45°)$. Since $\tan x° = \cot(90-x)° = 1/\tan(90-x)°$, the sum equals log 1 = 0.

F03S26. Let P be the northernmost point of the bridge and Q its southernmost point. The length of Matthew's path is $AP+PQ+QB=AP+QB+1$, which is a minimum when $AP+QB$ is minimum. Consider the translation of the plane 1 mile northward, and let B' be the image of B under this translation. Note that $AP+QB=AP+PB'$, which is a minimum when A, P and B' are collinear. Therefore, the minimum value of $AP+QB$ is $\sqrt{9^2+5^2}=\sqrt{106}$, and the path's minimum length is $1+\sqrt{106}$.

F03S27. Let M and L be Michael's and Lisa's ages now. When Michael was as old as Lisa is now, his age was L. Because that was M-L years ago, Lisa's age was $L-(M-L)=2L-M$, so $M=4(2L-M)$, and therefore $5M=8L$. When Michael was six years older than Lisa is now, Michael's age was $L+6$, and Lisa's age was $2L-M+6$, so $M=2(2L-M+6)$, and therefore $3M=4L+12$. Solve the system of equations to obtain M=24.

F03S28. Draw the four line segments joining the center of the sphere to the four vertices of the tetrahedron to form four smaller tetrahedral. Let B be the area of each face of the tetrahedron, let V be its volume and h its height, and let r be the radius of the sphere. The area of each of the four tetrahedral is $(1/3)Br$, so $(4/3)Br = V = (1/3)Bh$, and $4r = h$. To find h, consider the right triangle one of whose legs is the height from a vertex A, and whose hypotenuse is an edge with endpoint A. The length of the other leg is 2/3 the length of an altitude of a face-triangle, that is, $(2/3)3\sqrt{3}=2\sqrt{3}$. Thus $r=(1/4)h=(1/4)\sqrt{6^2-\left(2\sqrt{3}\right)^2}=\sqrt{6}/2.$

F03S29. Note that

$$\frac{\cos 15°+\sin 15°}{\cos 15°-\sin 15°}=\frac{\cos 15°+\sin 15°}{\cos 15°-\sin 15°}\cdot\frac{\cos 15°+\sin 15°}{\cos 15°+\sin 15°}=\frac{\cos^2 15°+\sin^2 15°+2\sin 15°\cos 15°}{\cos^2 15°-\sin^2 15°}$$

$$=\frac{1+\sin 30°}{\cos 30°}=\frac{3/2}{\sqrt{3}/2}=\sqrt{3}$$

F03S30. Solve a more general case. Let $AC=b$, $BC=a$, $AP=PQ=QB=x$, $CP=p$, and $CQ=q$. Use the median formulas to obtain $p^2=b^2/2+q^2/2-(2x)^2/4$ and $q^2=a^2/2+p^2/2-(2x)^2/4$, and add to obtain $p^2+q^2=\left(a^2+b^2\right)/2+\left(p^2+q^2\right)/2-2x^2$ $=\left(9x^2\right)/2+\left(p^2+q^2\right)/2-2x^2$. Thus $x^2=\left(p^2+q^2\right)/5=26$, and $AB=3x=3\sqrt{26}$.