NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Sophomore-Freshman Division

CONTEST NUMBER 1

PART I	*SPRING, 2003*	*CONTEST 1*	*TIME: 10 MINUTES*

S03SF1 Compute $27^{50} \div 81^{37}$.

S03SF2 A baseball is cut in half through the center. If the area of the circular region formed by the cut would be increased by 800%, then the volume of the baseball would be increased K%. Compute K.

PART II	*SPRING, 2003*	*CONTEST 1*	*TIME: 10 MINUTES*

S03SF3 The average of a and b is 13; the average of b and c is 31; and the average of c and a is 46. Compute the average of a, b, and c.

S03SF4 Start with 2 cups: Cup A with 100 mL of tea and Cup B with 100 mL of coffee. Take 2 mL from Cup A and mix it into Cup B. Then take 3 mL from Cup B and mix it into Cup A. Finally take 1 mL from Cup A and mix it into Cup B. If in the end Cup A has x mL of tea and Cup B has y mL of coffee, compute $x - y$.

PART III	*SPRING, 2003*	*CONTEST 1*	*TIME: 10 MINUTES*

S03SF5 Compute all values of x which satisfy

$$(x!)^2 = 25(x!) - 24$$

S03SF6 D is the midpoint of hypotenuse AC of right triangle ABC. Line segment BD is drawn. If $AD = \sqrt{29}$ and AB = 4, then compute the area of triangle BCD.

ANSWERS:		
	S03SF1	**9**
	S03SF2	**2600**
	S03SF3	**30**
	S03SF4	**0**
	S03SF5	**0, 1, 4**
	S03SF6	**10**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Sophomore-Freshman Division

CONTEST NUMBER 2

PART I — *SPRING, 2003* — *CONTEST 2* — *TIME: 10 MINUTES*

S03SF7 A lattice polygon is a polygon whose vertices all lie on points on a Cartesian plane with integer coordinates. Compute the shortest possible diagonal of a lattice rectangle whose area is 18 and whose sides are parallel to the x and y axes.

S03SF8 Alice has C cyan marbles, M magenta marbles, Y yellow marbles, and K black marbles. C, M, Y, and K are all greater than or equal to 0. It is noted that her marbles are all cyan except 3, all magenta except 3, and all yellow except 3. Compute all ordered quadruples (C,M,Y,K).

PART II — *SPRING, 2003* — *CONTEST 2* — *TIME: 10 MINUTES*

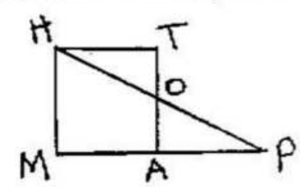
S03SF9 Which of the following is the smallest: 4^{12}, 5^{10}, or 6^8?

S03SF10 Find all positive integers n for which $n^5 - 1$ is prime.

PART III — *SPRING, 2003* — *CONTEST 2* — *TIME: 10 MINUTES*

S03SF11 16 teams entered a basketball tournament. If each team continues to play in the tournament until beaten twice, and the tournament ended with a champion in n games, compute all possible values of n.

S03SF12 The square MATH is 60 inches by 60 inches. HP is 156 inches and MAP and HOP are lines. Compute OA.



ANSWERS:

S03SF7	$3\sqrt{5}$
S03SF8	**(1,1,1,1), (0,0,0,3)**
S03SF9	6^8
S03SF10	**2**
S03SF11	**30, 31**
S03SF12	**35**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Sophomore-Freshman Division

CONTEST NUMBER 3

PART I *SPRING, 2003* *CONTEST 3* *TIME: 10 MINUTES*

S03SF13 The graduating class of Springfield Elementary averaged 87 on the final math exam. If the girls of that graduating class averaged a 91 and the boys in that class averaged an 81, compute the ratio of girls to boys in that class.

S03SF14 Compute the two numbers, x and y, whose sum, product, and quotient (x/y) are equal.

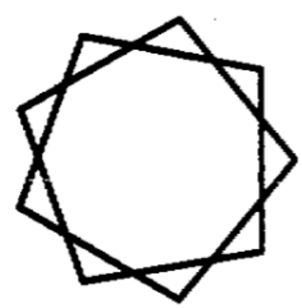
PART II *SPRING, 2003* *CONTEST 3* *TIME: 10 MINUTES*

S03SF15 The area of a non-degenerate circle ($r > 0$) is equal to the radius of the same circle. Compute the circumference of that circle.

S03SF16 How many <u>even</u> multiples of 7 are there between 1 and 1,000,000 that are also perfect squares?

PART III *SPRING, 2003* *CONTEST 3* *TIME: 10 MINUTES*

S03SF17 The sides of a regular nonagon (9-sided polygon) are extended to make a 9 pointed star as shown in the diagram. Compute the measure of the angle at a point of the star.



S03SF18 Find the greatest integer dividing 364, 414, and 539 with the same remainder in each case.

ANSWERS:		
	S03SF13	$\frac{3}{2}$
	S03SF14	$\frac{1}{2}, -1$
	S03SF15	**2**
	S03SF16	**71**
	S03SF17	**100**
	S03SF18	**25**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Freshman-Sophomore Division Solutions — Contest 1

S03SF1 $\frac{27^{50}}{81^{37}} = \frac{(3^3)^{50}}{(3^4)^{37}} = \frac{3^{150}}{3^{148}} = 3^2 = \mathbf{9}.$

S03SF2 The area is 9 times as big, so the radius is 3 times as big, implying that the volume is 27 times as big. Equivalently, the volume experienced a **2600%** increase.

S03SF3 $a + b + c = \frac{a+b}{2} + \frac{b+c}{2} + \frac{c+a}{2} = 13+31+46 = 90$

$\frac{a+b+c}{3} = \mathbf{30}$

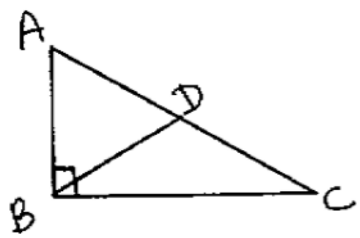
S03SF4 Note that the volume of the liquid in each of the 2 cups is still 100mL, so the amount of coffee in Cup A is $100 - x$. Which means that the amount of coffee in Cup B is $100 - (100 - x) = x = y$. The difference $x - y$ is **0**.

S03SF5 $(x!)^2 - 25(x!) + 24 = 0$

$((x!) - 24)((x!) - 1) = 0$

$x! = 24 \quad x! = 1$

$x = 4 \quad x = 0, 1$ The answer is **0, 1, 4**



S03SF6 $AC = 2\sqrt{29}$

$BC^2 + AB^2 = AC^2$

$BC^2 = 4^2 = (2\sqrt{29})^2$

$BC = 10$

The area of ΔABC is 20.

Area ΔABD + area ΔBCD = area ΔABC

However ΔABD and ΔBCD have equal bases (AD and DC) and the same height and therefore have the same area. Therefore area ΔBCD = (area ΔABC)/2 = **10**.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Freshman-Sophomore Division Solutions Contest 2

S03SF7 The possible length and widths of the rectangle are 1 and 18, 2 and 9, and 3 and 6. The one that provides the shortest diagonal is 3 and 6. The length of that particular diagonal is $\sqrt{3^2+6^2}=\sqrt{45}=3\sqrt{5}$.

S03SF8 Let the total number of marbles be T, then $T-3=C=M=Y$
$0 \le C$ and $2C = M + Y \le M + Y + K = T - C = 3$
C can be either 0 or 1, so the quadruples are **(1,1,1,1) and (0,0,0,3)**.

S03SF9 The fourth roots of these numbers are: 64, $25\sqrt{5}$, 36
Since $\sqrt{5} > 2$, $25\sqrt{5} > 50 > 36$. The smallest of them is $36^4 = \mathbf{6^8}$.

S03SF10 $n^5 - 1 = (n-1)(n^4+n^3+n^2+n+1)$
For $n = 1$, the product is 0
For $n = 2$, the product is 31 is prime
For $n > 2$, $n - 1 > 1$ and $n^4+n^3+n^2+n+1 > 1$ so the number is composite.
So the answer is **2**.

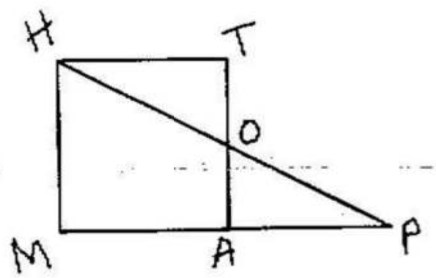
S03SF11 The champion can either be unbeaten or beaten once, so n can be $2\cdot 15$ or $2\cdot 15+1$
30, 31.

S03SF12 In right ΔHMP $60^2 + MP^2 = 156^2$

$MP^2 = 20736$

MP = 144

ΔHMP is similar to ΔOAP



$\frac{HM}{MP}=\frac{OA}{AP}$

$\frac{60}{144}=\frac{OA}{84}$

OA=35

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Freshman-Sophomore Division Solutions — Contest 3

S03SF13 Suppose there are G girls and B boys, then $91G + 81B = 87(G+B)$. So $4G = 6B$, or $\frac{G}{B} = \frac{3}{2}$

S03SF14 $xy = x/y$, so $xy^2 = x$ and $xy^2 - x = 0$, $x(y^2-1) = 0$ and $x = 0$ or $y = -1, 1$

Also $x + y = xy$. If $x = 0$, then $y = 0$ and x/y is undefined.

If $y = 1$, then $x + 1 = x$. This equation has no solution.

If $y = -1$, then $x - 1 = -x$. Therefore $x = 1/2$.

The answer is **1/2, -1**

S03SF15 Since $\pi r^2 = r$, we have $\pi r^2 - r = 0 = (\pi r - 1)(r)$
r cannot be 0, so r must be $1/\pi$
The circumference is $2\pi r =$ **2**

S03SF16 Any such number is of the form $2^2 7^2 k^2$ where k is a positive integer

Therefore $1 \le 2^2 7^2 k^2 \le 1{,}000{,}000$

Taking the square root gives $1 \le (2)(7)k \le 1000$

Dividing by 14 gives $1 \le k \le 71$. Therefore the answer is 71.

S03SF17 Extending the sides of the regular 9-gon produces 9 congruent isosceles triangles in its exterior. If we let x denote the measure of the base angle, then $9x = 360$, because the sum of the measures of the exterior angles of any n-gon is 360. So $x = 40$, and the measure of the angle at each point of the star is $180 - 40 - 40 =$ **100**.

S03SF18 Let d be the divisor, r be the remainder and q_1, q_2, q_3 be the quotients when d divides respectively 364, 414, and 539.

$414 = dq_2 + r$
$364 = dq_1 + r$

Subtracting gives $50 = dq_2 - dq_1 = d(q_2 - q_1)$. Therefore d divides 50.

$539 = dq_3 + r$
$414 = dq_2 + r$

Subtracting gives $125 = d(q_3 - q_2)$. Therefore d divides 125.

The greatest integer dividing 50 and 125 is 25. The answer is **25**.