New York City Interscholastic Mathematics League

SENIOR B DIVISION	**CONTEST NUMBER ONE**	**SPRING 2003**
PART I: 10 minutes	**NYCIML Contest One**	**Spring 2003**

S03B01. In Mr. Camel's class, the average age of the boys is 15, and the average age of the girls is 16. The ratio of the number of boys to girls is 3:2. Compute the average age of all the students in the class.

S03B02. Three fair dice are rolled, and the sum of the numbers shown is 7. Compute the probability that two dice show the same number.

PART II: 10 minutes	**NYCIML Contest One**	**Spring 2003**

S03B03. The number 7,973 can be factored as the product of 3 prime numbers. Compute the largest prime.

S03B04. Compute $\sin 75°$, in simplest radical form.

PART III: 10 minutes	**NYCIML Contest One**	**Spring 2003**

S03B05. If $x = 3^a + 7^b$, and a and b are non negative single digit integers, compute the probability that the units digit of x is 8.

S03B06. If the three numbers 796, 1157, and 1594 are divided by the positive integer q, they all leave a remainder of r. Compute r.

ANSWERS

1. **15.4**
2. $\frac{3}{5}$
3. **67**
4. $\frac{\sqrt{2}+\sqrt{6}}{4}$
5. $\frac{19}{100}$
6. **17**

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SENIOR B DIVISION **CONTEST NUMBER TWO** **SPRING 2003**

PART I: 10 minutes **NYCIML Contest Two** **Spring 2003**

S03B07. If $2^3 + 2^3 + 2^3 + 2^3 = 4^x$, compute x.

S03B08. The area of a rhombus is 100, and one diagonal is twice the other. Compute the length of a side of the rhombus.

PART II: 10 minutes **NYCIML Contest Two** **Spring 2003**

S03B09. Compute the sum of the digits of the first 100 positive integers.

S03B10. Compute all real values of x which satisfy

$$\log x + \log (x - 4) \le \log 221$$

PART III: 10 minutes **NYCIML Contest Two** **Spring 2003**

S03B11. Compute the number of positive integral divisors of 12^5

S03B12. The base of isosceles triangle ABC is $\overline{BC}$. D is a point on $\overline{AC}$ and E is a point on $\overline{AB}$. $AE = ED = DB = BC$. Compute $m\angle A$.

ANSWERS

7. $\frac{5}{2}$
8. $5\sqrt{5}$
9. 901
10. $4 < x \le 17$
11. 66
12. $\frac{180}{7}$

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SENIOR B DIVISION	**CONTEST NUMBER THREE**	**SPRING 2003**
PART I: 10 minutes	**NYCIML Contest Three**	**Spring 2003**

S03B13. A cube with edge 10 is painted, and cut into 1000 unit cubes. Compute the number of these unit cubes that are painted on exactly 2 sides.

S03B14. Compute the sum of the series $.7+.07+.007+.0007+\ldots$.

PART II: 10 minutes	**NYCIML Contest Three**	**Spring 2003**

S03B15. If $\log 2 = a$ and $\log 3 = b$, express $\log 750$ in terms of a and b with no other logarithms.

S03B16. If a, b, c are all nonzero numbers, compute all possible values of $\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}+\frac{ab}{|ab|}+\frac{ac}{|ac|}+\frac{bc}{|bc|}+\frac{abc}{|abc|}$.

PART III: 10 minutes	**NYCIML Contest Three**	**Spring 2003**

S03B17. If $x=\sqrt{3+\sqrt{3+\sqrt{3+\ldots}}}$, compute x.

S03B18. Until today's game, David had hit exactly 75% of his foul shots. Today he hit 3 out of 5 foul shots, and his percentage, when rounded off, was still 75%. Compute the fewest number of foul shots he could have taken this season, including today's game, for this to be possible.

ANSWERS

13. **96**
14. $\frac{7}{9}$
15. $b-2a+3$
16. **7,-1**
17. $\frac{1+\sqrt{13}}{2}$
18. **153**

New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER FOUR** **SPRING 2003**

PART I: 10 minutes **NYCIML Contest Four** **Spring 2003**

S03B19. Compute the units digit of $7^{2003} \cdot 9^{300}$.

S03B20. If $\log_8(\cos(x)) = -\frac{1}{2}$, compute $\cos(2x)$ with no logarithms.

PART II: 10 minutes **NYCIML Contest Four** **Spring 2003**

S03B21. If $\sqrt{x\sqrt{x\sqrt{x}}} = x^a$, compute a.

S03B22. Two roots of the equation $x^3 + ax^2 + 17x + b = 0$ are 1 and 2. Compute the third root.

PART III: 10 minutes **NYCIML Contest Four** **Spring 2003**

S03B23. Compute the area of a regular octagon which is inscribed in a circle of radius 10.

S03B24. Abel, Baker, and Charlie, in that order, each roll a pair of dice. The first person to roll a 7 wins. If no one rolls a 7 on the first round, they continue in the same order until someone wins. Compute the probability that Baker is the winner.

ANSWERS

19. **3**
20. $-\frac{3}{4}$
21. $\frac{7}{8}$
22. **5**
23. $200\sqrt{2}$
24. $\frac{30}{91}$

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **SPRING 2003**

PART I: 10 minutes **NYCIML Contest Five** **Spring 2003**

S03B25. $\frac{1}{4}, \frac{1}{x}$, and $\frac{1}{6}$, in that order, form an arithmetic progression. Compute x.

S03B26. Compute all real values of x which satisfy $2 \cdot 4^{2x} - 17 \cdot 4^{x} + 8 = 0$.

PART II: 10 minutes **NYCIML Contest Five** **Spring 2003**

S03B27. A penny, a nickel, a dime, a quarter, a half-dollar, and a dollar coin are on a table. Arthur picked up one or more coins. Compute how many different sums of money Arthur could have picked up.

S03B28. Compute the sum of all positive 2-digit integers that are not divisible by either 2 or 5.

PART III: 10 minutes **NYCIML Contest Five** **Spring 2003**

S03B29. If $3! \cdot 5! \cdot 7! = x!$, compute x.

S03B30. The legs of an isosceles triangle are of length 1 and the vertex angle is $36°$. Compute the length of the base.

ANSWERS

25. $\frac{24}{5}$

26. $\frac{3}{2}, -\frac{1}{2}$

27. 63

28. 1980

29. 10

30. $\frac{-1+\sqrt{5}}{2}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER ONE** **Spring 2003**

SOLUTIONS

S03B1. If there are $3x$ boys and $2x$ girls, $\dfrac{3x\cdot 15+2x\cdot 16}{5x}=\dfrac{77x}{5x}=\mathbf{15.4}$.

S03B2. The possibilities are:

(1,1,5) 3 ways (2,2,3) 3 ways

(1,2,4) 6 ways (1,3,3) 3 ways

Probablity = $\dfrac{9}{15}=\mathbf{\dfrac{3}{5}}$

S03B3. $7973 = 8000 - 27 = 20^3 - 3^3$. Since $a^3-b^3=(a-b)(a^2+ab+b^2)$,

$7973=(20-3)(20^2+3\cdot 20+3^2)=17\cdot 469=7\cdot 17\cdot \mathbf{67}$.

S03B4. $\sin 75^\circ=\sin(30^\circ+45^\circ)=\sin 30^\circ\cos 45^\circ+\cos 30^\circ\sin 45^\circ$

$$=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\mathbf{\frac{\sqrt{2}+\sqrt{6}}{4}}$$

S03B5. The units digit of the powers of 3 starting with exponent 0 are 1, 3, 9, 7, 1, 3, 9, 7, 1, 3. The units digit of the powers of 7 starting with exponent 0 are 1, 7, 9, 3, 1, 7, 9, 3, 1, 7. To get a unit's digit of 8, either

(9,9) 4 possibilities

(1,7) 9 possibilities

(7,1) 6 possibilities

P = $\mathbf{\dfrac{19}{100}}$

S03B6. Since they leave the same remainder, the difference between any two numbers must be a multiple of q.

$1157 - 796 = 361 = 19 \cdot 19$

$1594 - 1157 = 437 = 19 \cdot 23$

$q = 19$ and $r =$ **17**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER TWO** **Spring 2003**

SOLUTIONS

S03B7. $2^3+2^3+2^3+2^3=4\cdot 2^3=2^5=4^{5/2}$ The exponent is $\mathbf{\frac{5}{2}}$.

S03B8. The area of a rhombus is one half the product of the diagonals.

$\frac{1}{2}\cdot x\cdot 2x=x^2=100 \quad x=10 \qquad 2x=20$

In a rhombus, $s^2=\left(\frac{d_1}{2}\right)^2+\left(\frac{d_2}{2}\right)^2$, so $s^2=5^2+10^2$

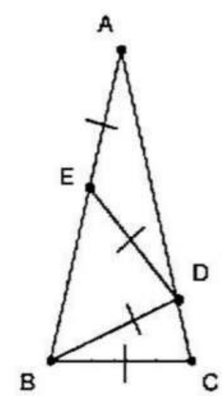
$s=5\sqrt{1+2^2}=\mathbf{5\sqrt{5}}$

S03B9. By symmetry, there are 10 of each digit in the 10's place and the unit's place. 10·45 + 10·45 = 900. Adding 1 for 100, the sum is **901**.

S03B10. The domain of the log function means x > 0 and x – 4 > 0.
Also log x(x – 4) ≤ log 221
So x(x – 4) ≤ 221
$x^2-4x-221\le 0$
(x – 17)(x + 13) ≤ 0 which gives a solution of –13 ≤ x ≤ 17
However the restricted domain causes the solution to be **4 < x ≤ 17**

S03B11. $12^5=2^{10}\cdot 3^5$. The number of factors is found by taking the product of 1 more than each exponent. Thus 11 · 6 = **66**.

S03B12. Let $\measuredangle A=x$
Then $\measuredangle ADE=x$
$\measuredangle DEB=2x$
$\measuredangle EBD=2x$
$\measuredangle EDB=180-4x$
$\measuredangle BDC=3x$
$\measuredangle C=3x$
$\measuredangle ABC=3x$
$x+3x+3x=180=7x$
$x=\frac{180}{7}$



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER THREE** **Spring 2003**

SOLUTIONS

S03B13. The cubes painted on 2 sides consist of the cubes on each of the 12 edges, except for the corners. $12 \cdot 8 =$ **96.**

S03B14. Solution 1: This is an infinite geometric progression:

$$S = \frac{9}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}$$

Solution 2: The number is .7777… which equals $\mathbf{\frac{7}{9}}$.

S03B15. $\log_{10} 750 = \log_{10}\left(3000/4\right) = \log_{10} 3 + 3\log_{10} 10 - \log_{10} 4 = \mathbf{b+3-2a}$

S03B16. If all 3 numbers are positive, the value is 7. If 1, 2, or all 3 numbers are negative, the value is -1.
ANS: **7, -1**

S03B17. $x = \sqrt{3+\sqrt{3+\sqrt{3+\ldots}}} = \sqrt{3+x}$

$x^2 = 3 + x \qquad x^2 - x - 3 = 0 \qquad x = \frac{1 \pm \sqrt{13}}{2}$

Rejecting the negative, $\mathbf{x = \frac{1+\sqrt{13}}{2}}$

S03B18. Let N be the number of shots David took until today's game.

$\frac{.75N+3}{N+5} \geq .745 \qquad .75N + 3 \geq .745(N+5)$

$750N + 3000 \geq 745N + 3725 \qquad N \geq 145$

Since N must be a multiple of 4, $N = 148$. Including today's game, he took **153** shots.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER FOUR** **Spring 2003**

SOLUTIONS

S03B19. The unit's digit of the powers of 7 run in cycles of 4, i.e. (7, 9, 3, 1). The powers of 9 runs in cycles of 2, i.e. (9, 1, 9, 1). $3 \cdot 1 = \mathbf{3}$

S03B20. $\cos x = 8^{-1/2} = \dfrac{1}{\sqrt{8}}$

$\cos 2x = 2\cos^2 x - 1 = 2 \cdot \dfrac{1}{8} - 1 = -\mathbf{\dfrac{3}{4}}$

S03B21. $\sqrt{x\sqrt{x\sqrt{x}}} = \sqrt{x\sqrt{x^{3/2}}} = \sqrt{x^{7/4}} = x^{7/8}$ The exponent is $\mathbf{\dfrac{7}{8}}$.

S03B22. 17 is the sum of the roots taken 2 at a time. $1 \cdot 2 + 2 \cdot r + 1 \cdot r = 17$
$r = \mathbf{5}$

S03B23. The isosceles triangle formed by 2 consecutive radii has a vertex angle of 45°. The area of each such triangle is $\dfrac{1}{2} \cdot 10 \cdot 10 \cdot \sin 45° = 25\sqrt{2}$.

$8 \cdot 25\sqrt{2} = \mathbf{200\sqrt{2}}$

S03B24. The probability of rolling a 7 is $\dfrac{1}{6}$. The probability of Baker winning on the first round is $\dfrac{5}{6} \cdot \dfrac{1}{6} = \dfrac{5}{6^2}$.

The probability that Baker will win on the second round is $\left(\dfrac{5}{6}\right)^4 \cdot \dfrac{1}{6} = \dfrac{5^4}{6^5}$

This is an infinite geometric progression.

$$S = \frac{\frac{5}{36}}{1 - \frac{125}{216}} = \mathbf{\frac{30}{91}}$$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION CONTEST NUMBER FIVE Spring 2003

SOLUTIONS

S03B25. $\frac{1}{x}$ is the average of $\frac{1}{4}$ and $\frac{1}{6}$. $\frac{1}{x}=\frac{\frac{1}{4}+\frac{1}{6}}{2}=\frac{5}{24}$

So $x=\mathbf{\frac{24}{5}}$

S03B26. Let $y=4^x$ $2y^2-17y+8=0$

$y=8$ or $y=\frac{1}{2}$

$x=\frac{3}{2}$ or $x=-\frac{1}{2}$ Answer: $\mathbf{\frac{3}{2},-\frac{1}{2}}$

S03B27. The number of subsets of 6 items is $2^6=64$. However, since the null set is excluded in the problem, the answer is **63**.

S03B28. This consists of 36 numbers, all the 2 digit numbers that end in 1,3,7, or 9

Solution 1:

$11+13+17+19$ The units digits in any row add up to 20.

$21+23+27+29$ $20\cdot 9+40+80+120+\ldots+360$

... $=180+1800=\mathbf{1980}$

Solution 2:

Adding the columns, the units digits add up to $9+27+63+81=180$

Each ten's column adds up to 450. $180+4\cdot 450=\mathbf{1980}$

Solution 3:

Notice that for every integer n that fits the above conditions, there exists a different and unique integer $110-n$ that also fits the above conditions. Since the pairings are distinct, there must be 18 pairs of such integers, each summing up to 110. The total is $18\cdot 110=\mathbf{1980}$.

S03B29. If the terms of $3!\cdot 5!$ Are rearranged, it equals $8\cdot 9\cdot 10$. $x=\mathbf{10}$.

S03B30. Bisect one of the base angles so that $\Delta CBD\sim\Delta ABC$

$\frac{1}{x}=\frac{x}{1-x}$

$x^2=1-x$ $x^2+x-1=0$

$x=\frac{-1\pm\sqrt{5}}{2}$

Rejecting the negative root, $x=\mathbf{\frac{-1+\sqrt{5}}{2}}$