NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division

CONTEST NUMBER 1

PART I ***SPRING, 2003*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S03S1 332812557 is a perfect cube. Compute the cube root.

S03S2 Compute the minimum value of $|x-1| - |x-2| + |x-3|$ for all real x.

PART II ***SPRING, 2003*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S03S3 The difference between the roots of $mx^2 + 7x - 12 = 0$ is 1.
Compute all real values of m.

S03S4 Compute the length of the shortest path that starts at (3,7), ends at (5,2), intersects the x-axis, the y-axis, and the line $y = x$.

PART III ***SPRING, 2003*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S03S5 Point P is 6 inches from the center of a circle with radius 10. Compute the number of chords with integral length that pass through P.

S03S6 Compute $\tan^2 10° + 2 \tan 10° \tan 70°$.

ANSWERS:

S03S1	**693**
S03S2	**1**
S03S3	**-1, 49**
S03S4	$\sqrt{145}$
S03S5	**8**
S03S6	**1**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior A Division

CONTEST NUMBER 2

PART I	*SPRING, 2003*	*CONTEST 2*	*TIME: 10 MINUTES*

S03S7 Compute all real values of x which satisfy

$$\log x + \log (x-4) \le \log 221$$

S03S8 Determine the real-valued polynomial function $f(x)$ that satisfies: $3f(x) - 5f(-x) = 2x^2 + 24x + 4$ for all real values of x.

PART II	*SPRING, 2003*	*CONTEST 2*	*TIME: 10 MINUTES*

S03S9 Compute the 2003rd number in the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6,…

S03S10 Compute all real values of x that satisfy: $\sqrt{x+9+2\sqrt{x+8}} + \sqrt{x+2-\sqrt{x+8}} = 4$.

PART III	*SPRING, 2003*	*CONTEST 2*	*TIME: 10 MINUTES*

S03S11 Compute the sum of a and b if the values of a and b make $x^4 - 24x^3 + 204x^2 + ax + b$ the square of a trinomial.

S03S12 Trapezoid MATH has bases $\overline{AT}$ and $\overline{MH}$. The diagonals $\overline{MT}$ and $\overline{AH}$ intersect at point P. The area of triangle PAT is 49 and the area of triangle MPH is 64. Find the area of trapezoid MATH.

ANSWERS:		
	S03S7	$4 < x \le 17$
	S03S8	$-x^2 + 3x - 2$
	S03S9	**63**
	S03S10	**1**
	S03S11	**180**
	S03S12	**225**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior A Division
CONTEST NUMBER 3

PART I	*SPRING, 2003*	*CONTEST 3*	*TIME: 10 MINUTES*

S03S13 If a, b and c are positive integers such that $3a + c = 4b$ and $2b = \sqrt{3ac}$. Compute the smallest possible value of $a + b + c$.

S03S14 The n-sectors (lines which divide an angle into n equal parts) of angles A and B of triangle ABC divide the interior of ABC into a certain number of non overlapping regions. Given that the number of non overlapping triangular regions is at most 25, compute the maximum possible number of non overlapping quadrilateral regions.

PART II	*SPRING, 2003*	*CONTEST 3*	*TIME: 10 MINUTES*

S03S15 Compute the number of ordered pairs of integers (x, y) such that $\sqrt{1008} = \sqrt{x} + \sqrt{y}$.

S03S16 If $a + b + c = 9$ and $a^3 + b^3 + c^3 = 99$, compute the value of $(a + b)(b + c)(c + a)$.

PART III	*SPRING, 2003*	*CONTEST 3*	*TIME: 10 MINUTES*

S03S17 Compute the number of ways 10 identical marbles can be placed in 3 different bags, if each bag must have at least two marbles?

S03S18 Compute all possible ordered triples (x, y, z) that satisfy the following equations.

$$\log_2 x \log_2 y + \log_2 xy = 2$$
$$\log_2 y \log_2 z + \log_2 yz = 59$$
$$\log_2 z \log_2 x + \log_2 zx = 4$$

ANSWERS:

S03S13	**11**
S03S14	**144**
S03S15	**13**
S03S16	**210**
S03S17	**15**
S03S18	$\left(\frac{\sqrt{2}}{2}, 32, 512\right), \left(\frac{\sqrt{2}}{4}, \frac{1}{128}, \frac{1}{2048}\right)$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division

CONTEST NUMBER 4

PART I	*SPRING, 2003*	*CONTEST 4*	*TIME: 10 MINUTES*

S03S19 In ΔABC, $BC = AB = 25$, $AC = 30$. Compute the length of the altitude from A to $\overline{BC}$.

S03S20 Compute all ordered pairs of positive integers that satisfy $x^2 + y^2 - xy = 49$.

PART II	*SPRING, 2003*	*CONTEST 4*	*TIME: 10 MINUTES*

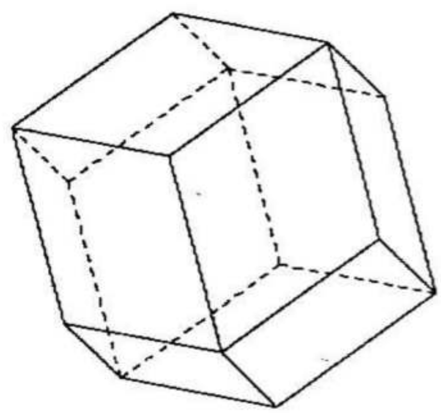
S03S21 Compute all x such that: $\left(x+1+x^{-1}\right)\left(x-1+x^{-1}\right)=\frac{21}{4}$.

S03S22 Compute the smallest number k such that if k elements are chosen from $\{1, 2, 3, \ldots, 143\}$, some three of these elements must be the sides of a non-degenerate triangle?

PART III	*SPRING, 2003*	*CONTEST 4*	*TIME: 10 MINUTES*

S03S23 If $x - y = 6$ and $xy = 4$, Compute the positive value of $x^3 + y^3$?

S03S24 A rhombus is formed from two isosceles triangles with sides 1, 1 and $2/\sqrt{3}$. A solid, known as the rhombic dodecahedron, has twelve faces, each of which is congruent to this rhombus. Compute the volume of this solid, given that multiple copies of it can fill space?



ANSWERS:

S03S19	**24**
S03S20	**(3,8), (5,8), (7,7), (8,3), (8,5)**
S03S21	$-2, -\frac{1}{2}, \frac{1}{2}, 2$
S03S22	**11**
S03S23	$80\sqrt{13}$
S03S24	$\frac{16\sqrt{3}}{9}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division

CONTEST NUMBER 5

PART I	*SPRING, 2003*	*CONTEST 5*	*TIME: 10 MINUTES*

S03S25 Solve for all real values of x: $(x+2)(x+3)(x-4)(x-5) = 120$.

S03S26 In the Cartesian plane, a circle is drawn centered at $C(-2, -5)$ with radius 8. The area of the part of the interior of the circle in quadrants I, II, III, and IV, are respectively A_1, A_2, A_3, and A_4. Compute the value of $A_1 - A_2 + A_3 - A_4$.

PART II	*SPRING, 2003*	*CONTEST 5*	*TIME: 10 MINUTES*

S03S27 Compute all real values of x that satisfy:

$$\log_2 \frac{2x-1}{x-2} < 0$$

S03S28 Given that n is an integer and that the ten's digit of n^2 is 7, compute the smallest possible unit's digit n^2 can have.

PART III	*SPRING, 2003*	*CONTEST 5*	*TIME: 10 MINUTES*

S03S29 A non-zero three-digit number $[A][B][C]$ in base 8 is equal to the three-digit number $[C][B][A]$ in base 7. Compute the number in base 10.

S03S30 Compute all θ in degrees such that $0° < \theta < 10°$ and

$$\sin(41\theta) = \sin(31\theta) + \cos(31\theta) + \cos(41\theta)$$

ANSWERS:

S03S25	**0, 2, $1 \pm 2\sqrt{6}$**
S03S26	**40**
S03S27	**$-1 < x < 1/2$**
S03S28	**6**
S03S29	**220**
S03S30	$\frac{5°}{2}, \frac{15°}{2}, 9°$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division Solutions — Contest 1

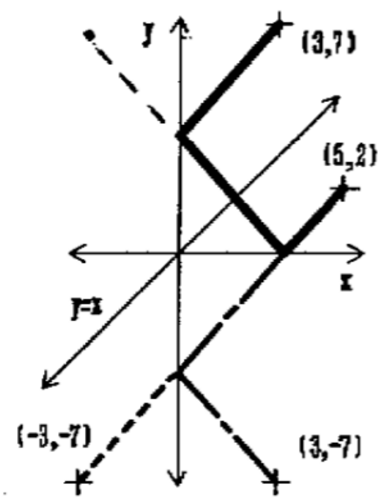
S03S1 Taken modulo 10, it is evident that only $3^3 \equiv 7 \pmod{10}$
$332812557 \approx 332000000 \approx 343000000 = 700^3$, so the hundred's digit is 6
Since 332812557 is divisible by 3, the cube root must also be divisible by 3. This means that the possible cube roots are 603, 633, 663, and 693. The closeness of 332 to 343 implies that the answer must be **693**. To check this result, let's cube **693** by hand:
$693^3 = (700-7)^3 = 343(100-1)^3 = 343\,(100^3 - 1) + 1029\,(100 - 100^2)$
$= [3][4-1][3][-2+1][-9][2][9-3][-4][-3] = 332812557$ (brackets represent digits)

S03S2 The graph of $|x-1| + |x-3|$ is flat in the interval [1,3] but slopes upward with slope ±2 everywhere else, so the minimum must exist somewhere in [1,3]. The graph of $-|x-2|$ in the interval [1,3] is lowest at points 1 and 3. So the minimum value of $|x-1| - |x-2| + |x-3|$ occurs at 1 or 3: $|1-1| - |1-2| + |1-3| = \mathbf{1}$.

S03S3 If p and q are the roots then $p - q = 1$, $p + q = -7/m$, and $pq = -12/m$.

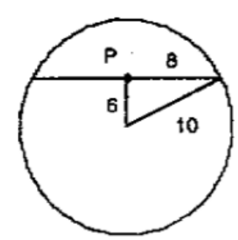
$$(p-q)^2 = (p+q)^2 - 4pq$$

$$1 = \frac{49}{m^2} + \frac{48}{m} \quad \text{Solving gives } m = -1, 49$$



S03S4 Since (5,2) and (3,7) are on different sides of $y = x$, any path between them will cross $y = x$. Since the two points are both to the right of the y-axis, the shortest path will touch the y-axis once and go right again. Reflecting the part of the path between the y-axis and (3,7) will have the two points on different sides of the line. The same can be done to the graph using the x-axis. So, the path in question is equivalent in length to the shortest path from (5,2) to (-3,-7), which by the Pythagorean Theorem, is $\sqrt{8^2+9^2} = \mathbf{\sqrt{145}}$.

S03S5 The shortest chord is perpendicular to the diameter through P, and has length 16. The longest is the diameter through P, and has length 20. There are 2 chords for each of the lengths 17, 18, and 19 through P, for a total of **8** chords.



S03S6 Let $A = 10°$, then the expression becomes $(\tan A)(\tan A + 2\tan(90°-2A))$.

$$(\tan A)(\tan A + 2\tan(90°-2A)) =$$

$$(\tan A)\left(\tan A + 2(\cot 2A)\right) = (\tan A)\left(\tan A + 2(\tan 2A)^{-1}\right)$$

$$= (\tan A)\left(\tan A + 2\left(\frac{1-\tan^2 A}{2\tan A}\right)\right) = (\tan A)\left(\tan A + \frac{1}{\tan A} - \tan A\right) = \mathbf{1}$$

New York City Interscholastic Mathematics League
Senior A Division Solutions Contest 2

S03S7 By the domain of a logarithmic function $x > 0$ and $x - 4 > 0$ so $x > 4$. Also $x(x-4) \le 221$. Solving this inequality yields $-13 \le x \le 17$. The intersection of the two sets of numbers is $\mathbf{4 < x \le 17}$.

S03S8 Replacing x with $-x$ in the equation $3f(x) - 5f(-x) = 2x^2 + 24x + 4$ gives $3f(-x) - 5f(x) = 2x^2 - 24x + 4$. We now have a system of two linear equations in $f(x)$ and $f(-x)$. Multiply the original equation by 3 and our new equation by 5, and add, to get $-16f(x) = 16x^2 - 48x + 32$. $\mathbf{f(x) = -x^2 + 3x - 2}$.

S03S9 If the n-th number in the sequence is k, then n is between $\frac{k(k-1)}{2}+1$ and $\frac{k(k+1)}{2}$, inclusive. Since $1024 = 2^{10} = 32^2$, 2048 must equal $64^2/2$. Therefore, $2048 - 32 = 2016$ marks the boundary between 63's and 64's, and $2048 - 32 - 63 = 1953$ would mark the boundary between 62's and 63's. 2003 is inside those bounds, so the 2003rd number in the sequence is 63

S03S10 Note that $\sqrt{x+9+2\sqrt{x+8}} = \sqrt{(x+8)+2\sqrt{x+8}+1}$, so $\sqrt{x+8}+1+\sqrt{x+2-\sqrt{x+8}} = 4$

The problem becomes: $3-\sqrt{x+8} = \sqrt{x+2-\sqrt{x+8}}$,

Squaring both sides gives $17 + x - 6\sqrt{x+8} = x + 2 - \sqrt{x+8}$,

and then $15 = 5\sqrt{x+8}$.

Solving gives $\mathbf{x = 1}$.

S03S11 $x^4 - 24x^3 + 204x^2 + ax + b = (x^2 + cx + d)^2$

$x^4 - 24x^3 + 204x^2 + ax + b = x^4 + 2cx^3 + (c^2 + 2d)x^2 + 2cdx + d^2$

Equating coefficients gives

$2c = -24,\quad c^2 + 2d = 204,\quad 2cd = a,\quad d^2 = b$

which gives $c = -12,\ d = 30,\ a = -720,\ b = 900$

Therefore $a + b = \mathbf{180}$.

S03S12 Let a = area of ΔMAP, b = area of ΔPTH. ΔMAP and ΔAPT have the same height h.

$$\frac{area\,\Delta MAP}{area\,\Delta APT} = \frac{a}{49} = \frac{\frac{1}{2}(MP)h}{\frac{1}{2}(PT)h} = \frac{MP}{PT} \quad \text{Similarly} \quad \frac{area\,\Delta MPH}{area\,\Delta PTH} = \frac{64}{b} = \frac{MP}{PT},\ so\ \frac{a}{49} = \frac{64}{b}.$$

ΔMAH and ΔMTH have the same base and height and therefore the same area.
So $a + 64 = b + 64$ and $a = b$. Therefore $a = b = (7)(8) = 56$.
Area of trapezoid MATH = $a + b + 64 + 49 = \mathbf{225}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division Solutions Contest 3

S03S13 put the two equations together: $3a + c = 4b = 2\sqrt{3ac}$

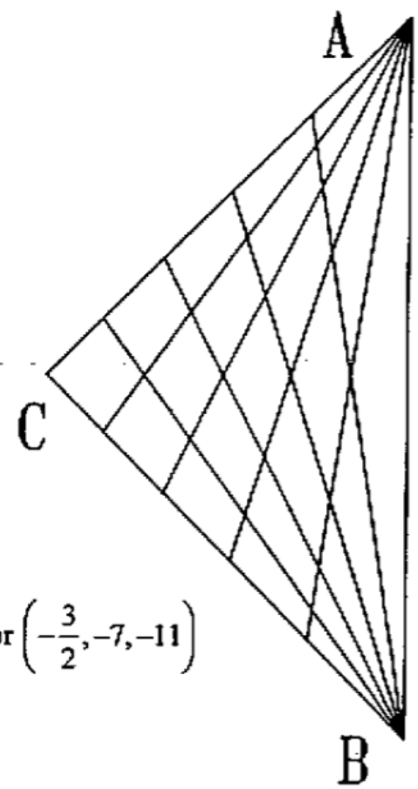
So $3a - 2\left(\sqrt{3a}\right)\left(\sqrt{c}\right) + c = 0 = \left(\sqrt{3a} - \sqrt{c}\right)^2$, which means $3a = c$

Substituting c for $3a$ in the first equation, we get $c = 2b$. The conditions $3a = c = 2b$ satisfies a minimum positive value when $a = 2$, $b = 3$, $c = 6$. Hence, the answer is **11**.

S03S14 There are n triangular regions each of whose vertices are at A, n triangular regions each of whose vertices are at B, one triangular region with vertices at both A and B, and no other triangular regions (the rest are all quadrilaterals). So $25 = 2n - 1$, $n = 13$. The number of quadrilateral regions is $(n-1)^2 = 12^2 =$ **144**.

S03S15 $\sqrt{y} = \sqrt{1008} - \sqrt{x}$ means $y = 1008 + x - 2\sqrt{1008x}$. It follows that y is an integer iff $1008x$ is a perfect square. $1008 = 12^2 \cdot 7$, hence x (and similarly y) must be of the form $7z^2$. Let $a = \sqrt{x}/\sqrt{7}$ and $b = \sqrt{y}/\sqrt{7}$, then the equation becomes $a + b = 12$ where a and b are non-negative integers. There are therefore **13** solutions.

S03S16 $729(a+b+c)^3 = a^3 + b^3 + c^3 + 3(ab^2 + ba^2 + bc^2 + cb^2 + ca^2 + ac^2 + 2abc)$ $= 99 + 3(a+b)(b+c)(c+a)$. This gives $3(a+b)(b+c)(c+a) = 630$, which means that $(a+b)(b+c)(c+a) =$ **210**.



S03S17 Six marbles are already accounted for, so there are only 4 marbles available for arbitrary distribution in the three bags. Call these marbles "arbitrary marbles". Suppose n arbitrary marbles are placed in bag 1, then there will be $4 - n$ marbles to share amongst bags 2 and 3, leading to $4 - n + 1$ choices. So the solution is: $\sum_{n=0}^{4}(5-n) = 5+4+3+2+1 = 15$.

S03S18 Let $\log_2 x = a$, $\log_2 y = b$, $\log_2 z = c$

then $ab + a + b + 1 = 2 + 1$ which becomes

$(a+1)(b+1) = 3$. Likewise $(b+1)(c+1) = 60$, $(c+1)(a+1) = 5$.

Multiply to obtain

$(a+1)^2(b+1)^2(c+1)^2 = 900$ and then $(a+1)(b+1)(c+1) = \pm 30$

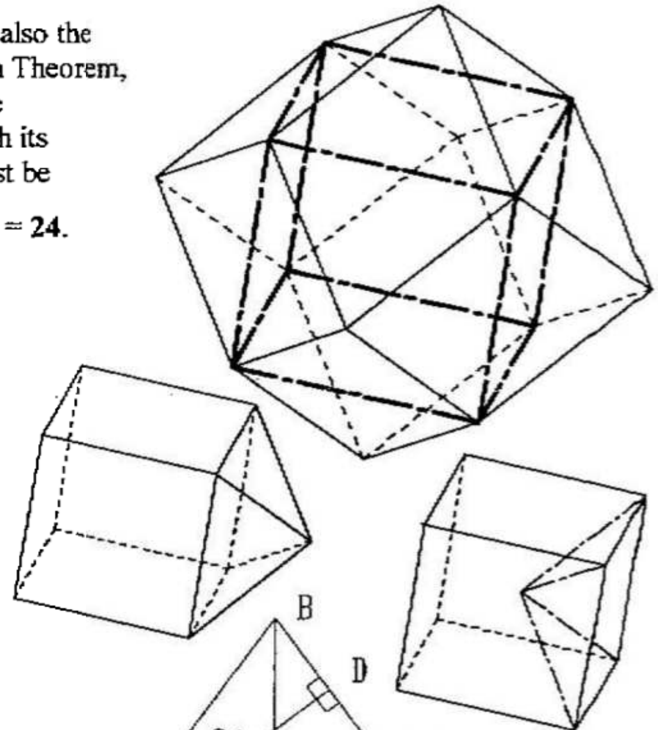
Therefore $a+1 = \frac{\pm 30}{60}$, $b+1 = \frac{\pm 30}{5}$, $c+1 = \frac{\pm 30}{3}$; $(a, b, c) = \left(-\frac{1}{2}, 5, 9\right)$ or $\left(-\frac{3}{2}, -7, -11\right)$

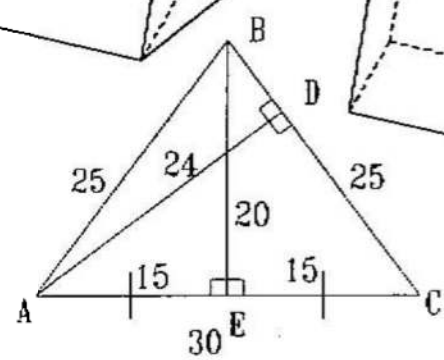
The answer for (x, y, z) is $\left(\frac{\sqrt{2}}{2}, 32, 512\right), \left(\frac{\sqrt{2}}{4}, \frac{1}{128}, \frac{1}{2048}\right)$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division Solutions — Contest 4

S03S19 ΔABC is isosceles, so the altitude to AC is also the median. That altitude, by the Pythagorean Theorem, is 20. Since the area of the triangle can be expressed as ½ the product of any side with its corresponding altitude, those products must be equal. So $AD = \dfrac{BE \cdot AC}{BC} = \dfrac{20 \cdot 30}{25}\ \dfrac{20 \cdot 30}{25} = \mathbf{24}$.





S03S20 Solving for y: $y = \dfrac{x \pm \sqrt{x^2 - 4\left(x^2 - 49\right)}}{2}$

So $196 - 3x^2$ must be an integer. Trying values for x from 1 to 8, it can be determined that the only solutions are: **(3,8), (5,8), (7,7), (8,3), and (8,5).**

S03S21 $\left(x + x^{-1}\right)^2 = \dfrac{25}{4}$, so $x + x^{-1} = \pm\dfrac{5}{2}$

$x^2 \pm \dfrac{5x}{2} + \dfrac{25}{16} = \dfrac{9}{16} = \left(x \pm \dfrac{5}{4}\right)^2$

$x = \pm\dfrac{5}{4} \pm \dfrac{3}{4}$ (the '±'s are independent)

The solutions are: $-2, -\frac{1}{2}, \frac{1}{2}, 2$

S03S22 The procedure must end in 11 choices. Assume for a contradiction, that the numbers $a_1 < a_2 < \ldots < a_{11}$ are chosen such that no triple of them forms a non-degenerate triangle. Then $a_3 \geq a_2 + a_1$, which makes $a_3 \geq 1 + 2 = 3$, $a_4 \geq 2 + 3 = 5$, $a_5 \geq 3 + 5 = 8$, $a_6 \geq 5 + 8 = 13$, $a_7 \geq 8 + 13 = 21$, $a_8 \geq 13 + 21 = 34$, $a_9 \geq 21 + 34 = 55$, $a_{10} \geq 34 + 55 = 89$, and $a_{11} \geq 55 + 89 = 144$, which is impossible. To see that the procedure doesn't necessarily end in 10 choices, consider the sequence of choices, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89; it's easy to see that no triple of them forms a non-degenerate triangle. So the answer is **11.**

S03S23 $(x + y)^2 = x^2 - 2xy + y^2 + 4xy = 36 + 16 = 52$

$x^3 + y^3 = (x + y)^3 - 3xy(x+y) = (x + y)\left((x + y)^2 - 3xy\right)$

$= \pm\sqrt{52}\,(52 - 3 \cdot 4) = \pm(40)(2)\sqrt{13}$ The positive solution is $80\sqrt{13}$.

S03S24 There are 2 types of vertices, one kind which joins 3 edges and another that joins 4. The set of all the 3 edge vertices form the vertices of a cube with side length $2/\sqrt{3}$. Truncate the polyhedron six times corresponding to the faces of said cube, and the remaining 6 pyramids will fit together to another cube of the same size. Thus the volume of the original polyhedron is twice that of the cube: $2\left(\dfrac{2}{\sqrt{3}}\right)^3 = \mathbf{\dfrac{16\sqrt{3}}{9}}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior A Division Solutions — Contest 5

S03S25 $[(x+2)(x-4)]\,[(x+3)(x-5)] = 120$ and then $(x^2-2x-8)(x^2-2x-15)=120$
Let $u = x^2-2x$ Then you have $(u-8)(u-15)=120$. Solving gives $u=0$ or $u=23$.
Solving $u = x^2-2x = 0$ and $x^2-2x = 23$ gives $x = 0,\ 2,\ 1\pm2\sqrt{6}$

S03S26 Divide up the interior of the circle into 9 parts using the x and y axes and the reflections of the axes with respect to point C ($x=-4$ and $y=-10$). Order and name the regions top down then from left to right as $R_1, R_2, \ldots, R_9$. Notice that $R_1 = R_9 = R_3 = R_7$, $R_2 = R_8$, and $R_4 = R_6$.
So $A_1 - A_2 + A_3 - A_4 = R_3 - R_1 - R_2 + R_4 + R_5 + R_7 + R_8 - R_6 - R_9 =$
$R_1 - R_1 - R_2 + R_4 + R_5 + R_1 + R_2 - R_4 - R_1 = R_5$. The area of R_5 is $4\cdot 10 = \mathbf{40}$.

S03S27 By the domain of a log function

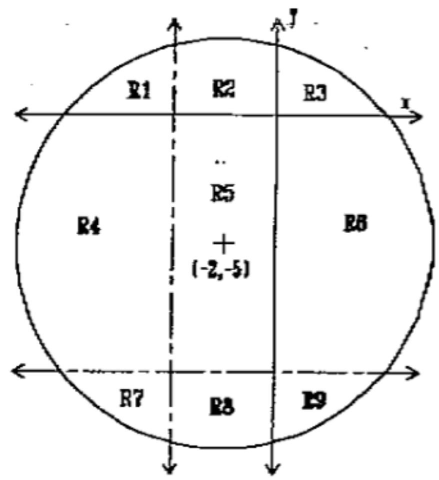
$\frac{2x-1}{x-2} > 0 \ so \ x < \frac{1}{2} \ or \ x > 2$

However $\log_2 \frac{2x-1}{x-2} < 0 \ so \ \frac{2x-1}{x-2} < 2^0$ and

$\frac{2x-1}{x-2} - 1 < 0 \ so \ \frac{2x-1}{x-2} - \frac{x-2}{x-2} < 0$

$\frac{x+1}{x-2} < 0 \ so \ -1 < x < 2$ Therefore the intersection of

the two restricted solutions gives $-1 < x < \frac{1}{2}$



S03S28 Squares are 0 or 1 modulo 4: 72, 73, 76, or 77
Squares are 0, 1, or 4 modulo 5: 70, 71, 74, 75, 76, 79
Only 76 fits both criteria, so the last digit is **6**.

S03S29 $64A + 8B + C = 49C + 7B + A$
$B = 48C - 63A = 3(16C - 21A)$
B is divisible by 3, so B is either 0, 3, or 6
Corresponding A values that will make $21A+B/3$ divisible by 16 are: 0, 3, 6
Corresponding C values are: 0, 4, 8
The only non-zero solution in base 7 is $334_8 = 433_7 = \mathbf{220}$

S03S30 We have $\sin(41\theta) - \sin(31\theta) = \cos(31\theta) + \cos(41\theta)$
Let $A = 36\theta$, $B = 5\theta$. Then $\sin(A+B) - \sin(A-B) = \cos(A-B) + \cos(A+B)$
which yields $2\sin B\cos A = 2\cos A\cos B$
then $2\cos A(\sin B - \cos B) = 0$ and $\cos A = 0$ or $\tan B = 1$
Now $0° < A < 360°$ and $0° < \theta < 50°$ since $A = 36\theta$, $B = 5\theta$ and $0° < \theta < 10°$.
Thus $A = 36\theta = 90°$ or $270°$ and $B = 5\theta = 45°$.
Therefore $\theta = \frac{90°}{36}, \frac{270°}{36}, \frac{45°}{5}$ and the answer is $\theta = \frac{5°}{2}, \frac{15°}{2}, 9°$