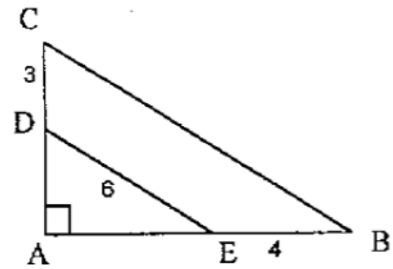
New York City
Interscholastic
Mathematics
League

JUNIOR DIVISION **CONTEST NUMBER ONE** **SPRING 2003**

PART I: 10 minutes **NYCIML Contest One** **Spring 2003**

S03J1. Compute any prime factor of 999973.

S03J2. Line segment $\overline{DE}$ is parallel to side $\overline{BC}$ of right triangle ABC. CD = 3, DE = 6, and EB = 4. Compute the area of quadrilateral $BCDE$.



PART II: 10 minutes **NYCIML Contest One** **Spring 2003**

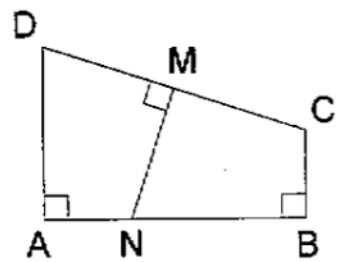
S03J3. $\dfrac{2x^2+7x+3}{2x^2-x-1}=\dfrac{1}{2}$. Compute x.

S03J4. The graph of the equation $y=ax^2+bx+c$ is a parabola that passes through the points $(5,3)$ and $(-2,-4)$. If a, b, and c are integers, compute the smallest positive value b can have.

PART III: 10 minutes **NYCIML Contest One** **Spring 2003**

S03J5. Compute how many integers x, $1 \le x \le 500$, are __not__ divisible by either 3 or 7.

S03J6. In trapezoid $ABCD$, leg $\overline{AB}$ is perpendicular to bases $\overline{AD}$ and $\overline{BC}$. $\overline{MN}$, the perpendicular bisector of $\overline{CD}$, divides the trapezoid into two regions of equal area. If $AN=1$ and $NB=2$, compute MN.



ANSWERS:

S03J1. **97 or 13 or 61 (only one prime is needed)**

S03J2. **102/5 or 20.4 or** $20\frac{2}{5}$

S03J3. **–7**

S03J4. **4**

S03J5. **286**

S03J6. $\dfrac{\sqrt{10}}{2}$

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JUNIOR DIVISION — **CONTEST NUMBER TWO** — **SPRING 2003**

PART I: 10 minutes — **NYCIML Contest Two** — **Spring 2003**

S03J7. Compute the number of diagonals that can be drawn in a convex 13-sided polygon.

S03J8. Two tokens, a and b, are drawn at random from a bag containing tokens $1, 2, \ldots, 40$. a is revealed and b is hidden. It is noted that the probability that b is a factor or a multiple of a is $\frac{1}{13}$. Compute the number of possible values of a.

PART II: 10 minutes — **NYCIML Contest Two** — **Spring 2003**

S03J9. Compute the ordered pair of positive integers (m, n), satisfying $3m^2 - 37n^2 + 1 = 0$ for which m is the smallest.

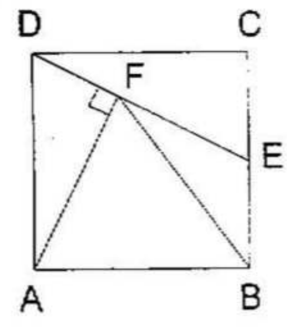
S03J10. Compute all ordered triples (x, y, z) of integers such that:

$$x + y = 6$$
$$x^2 + z^2 = 25$$
$$y^3 + z^3 = 91$$

PART III: 10 minutes — **NYCIML Contest Two** — **Spring 2003**

S03J11. Compute the number of distinct scalene triangles such that all of the side lengths are elements of the set $\{6, 12, 14, 19, 24\}$.

S03J12. Square $ABCD$ has an integral side length. E is the midpoint of $\overline{BC}$, and $\overline{AF} \perp \overline{DE}$. If the area of ΔBEF is an integer, compute the minimum value of AB.



ANSWERS:

S03J7. **65**
S03J8. **11**
S03J9. **(7, 2)**
S03J10. **(3, 3, 4) and (0, 6, -5)**
S03J11. **7**
S03J12. **10**

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JUNIOR DIVISION | **CONTEST NUMBER THREE** | **SPRING 2003**

PART I: 10 minutes | **NYCIML Contest Three** | **Spring 2003**

S03J13. If 2 fair, standard, six-sided dice are rolled, compute the probability that the absolute value of the difference of the numbers facing up is less than or equal to one.

S03J14. If $z+\frac{1}{z}=5$, compute $\left|z^2-\frac{1}{z^2}\right|$.

PART II: 10 minutes | **NYCIML Contest Three** | **Spring 2003**

S03J15. Compute the number of ordered pairs (x, y) of positive integers such that $x+y=696$ and the greatest common divisor of x and y is 58.

S03J16. A class has 34 students. 17 students know how to play chess, 8 know how to play bridge, and 26 know how to play hearts. If only 1 student knows how to play all three games, but every student knows how to play at least one game, compute how many students know how to play exactly two of the three games.

PART III: 10 minutes | **NYCIML Contest Three** | **Spring 2003**

S03J17. Compute all real values of x which satisfy
$$\sqrt{x+\sqrt{x+47}}+\sqrt{x-\sqrt{x+47}}=8$$

S03J18. A stack of magazines is composed of six different issues of "Mathy Mayhem". Two of the issues have blue covers, two have gray covers, one has a green cover and one has a red cover. Compute the number of ways the magazines can be ordered so that no issue lies directly on top of an issue with the same color cover.

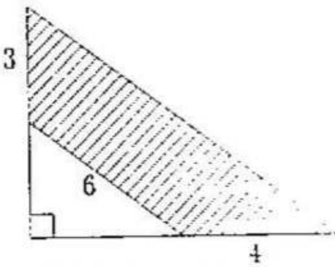
ANSWERS:

S03J13. $\frac{4}{9}$
S03J14. $5\sqrt{21}$
S03J15. **4**
S03J16. **15**
S03J17. **17**
S03J18. **336**

New York City Interscholastic Mathematics League

JUNIOR DIVISION **CONTEST NUMBER ONE SOLUTIONS** **SPRING 2003**

S03J1. $999973 = 1000000 - 27 = 100^3 - 3^3 = \mathbf{97}\,(10000 + 300 + 9) = \mathbf{97}\,(10309) = \mathbf{97}(\mathbf{13^2})(\mathbf{61})$
Note: Finding **ONE** prime factor is enough. Students are not expected to find more than one.



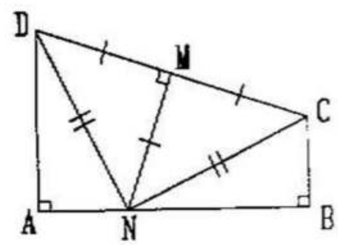
S03J2. Draw 2 lines perpendicular to the diagonal segment marked with length 6, one at each endpoint of the said segment. The shaded figure is split in to three regions by these 2 lines. The top and bottom regions combine to form a right triangle with legs of 3 and 4, while the middle region is a rectangle with length 6 and width equaling to the height of that triangle drawn to the hypotenuse. The area of that triangle is therefore 6 and the height measured to the hypotenuse is therefore 12/5. The area of the shaded region is 6 + 72/5 = 102/5.

S03J3. $\frac{1}{2} = \frac{2x^2+7x+3}{2x^2-x-1} = \frac{(2x+1)(x+3)}{(2x+1)(x-1)} = \frac{x+3}{x-1}$ if $x \neq -\frac{1}{2}$. (But if $x = -\frac{1}{2}$, the expression isn't defined anyway). So $x - 1 = 2x + 6$, and $x = \mathbf{-7}$.

S03J4. $3 = 25a + 5b + c$ and $-4 = 4a - 2b + c$, so $7 = 21a + 7b = 7(3a+b)$, so $3a + b = 1$, and the minimum positive value of b would be 1 if a could be 0. But that would make the graph a line, not a parabola. $b = 2$ and $b = 3$ don't make a integral. $b = 4$ makes $a = -1$, so $\mathbf{b = 4}$ is in fact the minimum.

S03J5. The number of integers between 1 and 500, inclusive, that are divisible by 3 or 7, is $\left[\frac{500}{3}\right] + \left[\frac{500}{7}\right] - \left[\frac{500}{21}\right] = 214$. So the answer is $500 - 214 = \mathbf{286}$.

S03J6. (Brackets denote area). Since $[DMNA] = [NMCB]$ (given) and $[DMN] = [CMN]$ (because $\overline{MN}$ is the perpendicular bisector of $\overline{CD}$), $[DAN] = [NBC]$. So triangles DAN and NBC are right triangles with congruent hypotenuses and equal areas, and they must be congruent. (To see this, inscribe them in the same semicircle; since their areas are equal, their heights to the diameter – their common hypotenuse – must be equal, so they are either the same triangle or they're reflections of each other about the perpendicular bisector of the diameter). Therefore, DNC is a right angle, so DNC is a 45°–45°–90° triangle, because $DN = NC$. Therefore:



$$MN = MC = \frac{NC}{\sqrt{2}} = \frac{\sqrt{NB^2+BC^2}}{\sqrt{2}} = \frac{\sqrt{NB^2+AN^2}}{\sqrt{2}} = \frac{\sqrt{4+1}}{\sqrt{2}} = \frac{\sqrt{\mathbf{10}}}{\mathbf{2}}.$$

New York City Interscholastic Mathematics League

JUNIOR DIVISION **CONTEST NUMBER TWO SOLUTIONS** **SPRING 2003**

S03J7. For an n-sided figure, there are exactly n – 3 diagonals from any vertex. Since there are n vertices, there are $n\cdot(n-3)$ positions where diagonals would meet vertices. Since diagonals have 2 endpoints, a division by 2 is needed: $\frac{n\cdot(n-3)}{2}$. $\frac{13\cdot(13-3)}{2}=\mathbf{65}$.

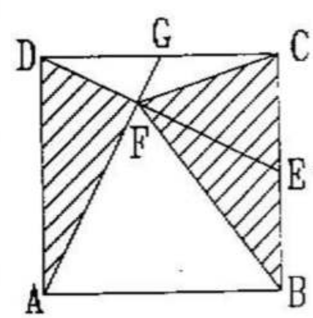
S03J8. Once a is known, there are 39 possible values for b. Since $\frac{1}{13}=\frac{3}{39}$, it follows that there are exactly 3 integers (different from a) among 1, 2, ..., 40 that are factors or multiples of a. If a were composite and not a perfect square of a prime (4, 9, 25 don't work), then it would have at least 3 factors other than itself, so it would indeed have to have *exactly* 3 such factors (4 divisors altogether), and no multiples greater than it among 1, 2, ..., 40. It would then have to be > 20. Now, the numbers with 4 divisors are the ones of the form pq or p^3 (p and q denote primes). The numbers > 20 and ≤ 40 that are of one of these forms, are 27, $2\cdot 11$, $2\cdot 13$, $2\cdot 17$, $2\cdot 19$, $3\cdot 7$, $3\cdot 11$, $3\cdot 13$, and $5\cdot 7$. If a were prime, on the other hand, it would have exactly 1 factor less than it, and would thus need to have exactly 2 multiples greater than it and ≤ 40. The primes satisfying this condition are 13 and 11. So there's a total of 9 + 2 = **11** possible values of a.

S03J9. $m^2=\frac{37n^2-1}{3}$, so n must be of the form $3k\pm 1$. n = 1 doesn't give a square when substituted into the expression, but n = 2 gives the square 49. The answer is then **(7, 2)**.

S03J10. The only pairs of squares that add up to 25 are {9, 16} and {0, 25}. Therefore, $z^3=\pm 3^3$, $\pm 4^3$, 0, or $\pm 5^3$, so y^3 has possible values: 91+27=118, 91-27=64, 91+64=155, 91-64=27, 91+125=216, 91-125= -34, or 91. The only ones of these that are cubes, though, are 64, 27, and 216. So (y, z) = (4, 3), (3, 4), or (6, -5). The second equation makes $(x,y,z)=(\pm 4,4,3)$, $(\pm 3,3,4)$, or (0, 6, -5). The only ones of these that also satisfies the first equation are **(3, 3, 4) and (0, 6, -5)**.

S03J11. If $a < b < c$, then it suffices to have $c < a + b$ in order that the three values be the lengths of the sides of a triangle. So, if 14 is the largest side there's exactly 1 triangle, if 19 is the largest there are 2 triangles (given by {6, 14, 19} and {12, 14, 19}), and if 24 is the largest there are 4 triangles (given by {6, 19, 24}, {12, 14, 24}, {12, 19, 24}, {14, 19, 24}). The answer is thus 1 + 2 + 4 = **7**.

S03J12. Let the side of the square be n, then [ADF] + [BCF] = $\frac{1}{2}n^2$. (Brackets denote area.) Let the [BEF] be k, then $2k$ = [BCF]. Extend $\overline{AF}$ to G on $\overline{CD}$, notice that $\Delta AFD \sim\Delta\ ADG \sim\Delta\ DCE$, so $\tan GAD=\frac{1}{2}$ and $\frac{AD}{AG}=\frac{2}{\sqrt{5}}$. $[AFD]=\frac{4}{5}[ADG]=\frac{4}{5}\cdot\frac{1}{4}[ABCD]=\frac{1}{5}n^2$. Substituting for [$ADF$] and [$BCF$]: $\frac{1}{5}n^2+2k=\frac{1}{2}n^2$. Or $k=\frac{3}{20}n^2$. The minimum integral value of k is 15. This gives n = **10**.



JUNIOR DIVISION **CONTEST NUMBER THREE SOLUTIONS** **SPRING 2003**

S03J13. If the value of the first die is n, then the value of the second die can be n-1, n, or n+1. n-1 cannot be obtained only when n=*1*. n+1 cannot be obtained only when n=6. So the probability is $\frac{3\cdot 6-2}{36}=\frac{16}{36}=\frac{\mathbf{4}}{\mathbf{9}}$.

S03J14. $z^2+\frac{1}{z^2}=(z+\frac{1}{z})^2-2\cdot z\cdot\frac{1}{z}=25-2=23$

so $(z-\frac{1}{z})^2=z^2+\frac{1}{z^2}-2\cdot z\cdot\frac{1}{z}=23-2=21$

Therefore, $|z^2-\frac{1}{z^2}| = |(z-\frac{1}{z})(z+\frac{1}{z})| = |z-\frac{1}{z}|\cdot|z+\frac{1}{z}| = \sqrt{21}\cdot 5=\mathbf{5\sqrt{21}}$.

S03J15. Since GCD(x, y) = 58, we can write $x = 58X$ and $y = 58Y$, where GCD(X, Y) = 1. Then $58X + 58Y = 696$, so $X + Y = 12$. The pairs (X, Y) with GCD(X, Y) = 1 that work are (1, 11), (5, 7), (7, 5), (11, 1). So the number of solutions is **4**.

S03J16. Let C denote chess, N(C) the number of chess players, N(CH) the number of people who play C and H, etc. Exclude the one person who plays all three games. The result is a class of 33 students with 16 chess players, 7 bridge players, and 25 hearts players, and contains only the students who played 2 games or less. In this new class, 33 = N(C) + N(B) + N(H) – N(CB) – N(CH) – N(BH) = 16 + 7 + 25 – (the number of students who play exactly two games). So the answer is 16 + 7 + 25 – 33 = **15**.

S03J17. Square both sides: $x+\sqrt{x+47}+2\sqrt{x^2-x-47}+x-\sqrt{x+47}=64$

$$\sqrt{x^2-x-47}=32-x$$
$$x^2-x-47=1024-64x+x^2$$
$$63x=1071$$
$$x=17$$

S03J18. The number of ways to arrange the magazines with no restrictions is 6! = 720. The number of arrangements where the two blue magazines are together (or where the two gray magazines are together) is $2\cdot 5!=240$. The number of arrangements where the two blue magazines are together *and* the two gray magazines are together is $2\cdot 2\cdot 4!=96$. So the number of arrangements where neither the gray nor the blue magazines are together is $720-2\cdot 240+96$ = **336**.