New York City Interscholastic Mathematics League

SENIOR B DIVISION **CONTEST NUMBER ONE** **FALL 2002**

PART I: 10 minutes **NYCIML Contest One** **Fall 2002**

F02B01. Compute the number of 3 letter arrangements that can be formed if no letter is repeated and the letters are in alphabetical order. (ABC is one example.)

F02B02. Compute the area of a triangle whose sides are of integral length and whose perimeter is 8.

PART II: 10 minutes **NYCIML Contest One** **Fall 2002**

F02B03. Compute the sum of all positive integers greater than 1 and less than 1000, which leave a remainder of one when, divided by either 2,3,4,5, or 6.

F02B04. Three positive integers form an arithmetic progression. If the smallest number is increased by one, the arithmetic progression becomes a geometric progression. In the original arithmetic progression if the largest number is increased by two, the arithmetic progression also becomes a geometric progression. Compute the smallest number of the original arithmetic progression.

PART III: 10 minutes **NYCIML Contest One** **Fall 2002**

F02B05. Compute the coefficient of x^3 when $(2x-3)^6$ is expanded.

F02B06. If x represents a two-digit positive integer, compute the number of values of x such that x^2+3x+2 is divisible by 6.

<u>ANSWERS</u>

1. **2600**
2. $\mathbf{2\sqrt{2}}$
3. **8176**
4. **8**
5. **−4320**
6. **60**

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SENIOR B DIVISION **CONTEST NUMBER TWO** **FALL 2002**

PART I: 10 minutes **NYCIML Contest Two** **Fall 2002**

F02B7. Compute: $\log_2\left(\frac{128\sqrt{8}}{\sqrt[3]{32}}\right)$.

F02B8. If p and q are the roots of the equation $ax^2 + bx + c = 0$, express $\frac{1}{p^2}+\frac{1}{q^2}$ in simplest form in terms of a, b, and c.

PART II: 10 minutes **NYCIML Contest Two** **Fall 2002**

F02B9. Compute the sum of the coefficients in the expansion of $(x - y)^{10}$.

F02B10. The medians to the legs of a right triangle measure $\sqrt{37}$ and $\sqrt{38}$. Compute the length of the hypotenuse of this triangle.

PART III: 10 minutes **NYCIML Contest Two** **Fall 2002**

F02B11. Compute all values of x such that $(x+3)^{x^2-5x+6}=1$

F02B12. P is a point inside rectangle $ABCD$. If PA, PB and PC are 4, 6 and 8 respectively, compute PD.

ANSWERS

7. $\frac{41}{6}$ or $6\frac{5}{6}$
8. $\frac{b^2 - 2ac}{c^2}$
9. 0
10. $2\sqrt{15}$
11. -4, -2, 2, 3
12. $2\sqrt{11}$

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SENIOR B DIVISION **CONTEST NUMBER THREE** **FALL 2002**

PART I: 10 minutes **NYCIML Contest Three** **Fall 2002**

F02B13. Convert the number 109 from base 10 to base 2.

F02B14. Compute the number of ordered pairs of positive integers (x, y) such that $2x + 3y = 2002$.

PART II: 10 minutes **NYCIML Contest Three** **Fall 2002**

F02B15. Compute the sum of all 2 digit even integers.

F02B16. Compute the sum: $\frac{1}{5}+\frac{2}{5^2}+\frac{1}{5^3}+\frac{2}{5^4}+\frac{1}{5^5}+\frac{2}{5^6}+\ldots$

PART III: 10 minutes **NYCIML Contest Three** **Fall 2002**

F02B17. If $\left(x+\frac{1}{x}\right)^2=3$, compute $x^3+\frac{1}{x^3}$.

F02B18. Compute the number of terms in the expansion of $(x+y+z)^{10}$.

ANSWERS

13. 1101101_2 or 1101101
14. 333
15. 2430
16. $\frac{7}{24}$
17. 0
18. 66

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SENIOR B DIVISION **CONTEST NUMBER FOUR** **FALL 2002**

PART I: 10 minutes **NYCIML Contest Four** **Fall 2002**

F02B19. If $\log_2(\log_3(\log_4 x)) = 0$, compute x.

F02B20. A machine can sort a bag of mail in 20 minutes; a postal worker can sort the same mail in 30 minutes. Compute the number of minutes it would take 3 postal workers and 4 machines working together to sort a bag of mail?

PART II: 10 minutes **NYCIML Contest Four** **Fall 2002**

F02B21. A square of area 10 is inscribed in a semicircle S. Compute the area of a square that is inscribed in a circle P that has the same radius as the semicircle S.

F02B22. Compute the number of zeroes at the end of 50!

PART III: 10 minutes **NYCIML Contest Four** **Fall 2002**

F02B23. Joe's average on his 9 math tests was 82. However, he forgot that his teacher would not count his lowest grade, a 50. Compute his average on the remaining 8 tests.

F02B24. To the nearest second, the first time of the day that the hour hand and the minute hand of a clock form a right angle is x minutes and y seconds after midnight. Compute the ordered pair (x, y).

ANSWERS

19. **64**
20. $\mathbf{\frac{10}{3}}$
21. **25**
22. **12**
23. **86**
24. **(16, 22)**

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SENIOR B DIVISION **CONTEST NUMBER FIVE** **FALL 2002**

PART I: 10 minutes **NYCIML Contest Five** **Fall 2002**

F02B25. Compute: $\dfrac{600^2}{301^2-299^2}$.

F02B26. Compute the number of positive integral divisors of 2002.

PART II: 10 minutes **NYCIML Contest Five** **Fall 2002**

F02B27. Three balls are chosen without replacement from 15 balls numbered 1 through 15. Compute the probability that the product of the numbers on the 3 balls picked is odd.

F02B28. Compute the volume of a regular tetrahedron with edge of length 3.

PART III: 10 minutes **NYCIML Contest Five** **Fall 2002**

F02B29. Triangle ABC is equilateral, with side of length 10. If $\overline{BD}$ and $\overline{CE}$ are medians that meet at F, compute the area of triangle BCF.

F02B30. Compute the ordered pair of integers (a, b) if the base 10 number $\underline{32a594601b71}$ is divisible by 99. (a and b are digits of the 12 digit base 10 number)

ANSWERS

25. **300**
26. **16**
27. $\mathbf{\dfrac{8}{65}}$
28. $\dfrac{9}{4}\sqrt{2}$
29. $\dfrac{25}{3}\sqrt{3}$
30. **(2, 5)**

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER ONE** **Fall 2002**

SOLUTIONS

F02B1. The number of ways to take 3 letters from 26 is ${}_{26}C_3 = 2600$. Since each one will produce exactly 1 arrangement in alphabetical order, the answer is **2600**.

F02B2. The only triangle possible is an isosceles triangle with legs 3 and base 2. $K = \frac{1}{2} * 2 * 2\sqrt{2} = \mathbf{2\sqrt{2}}$

F02B3. All numbers that work must be one more than a multiple of 60, from 61 to 961. The sum of these 16 numbers is $\frac{16(61+961)}{2} = \mathbf{8176}$.

F02B4.

Let the integers be a-b, a, and $a+b$

Then $a^2 = (a-b+1)(a+b) = a^2 - b^2 + (a+b)$ $\quad$ $a^2 = (a-b)(a+b+2) = a^2 - b^2 + 2(a-b)$

$b^2 = a + b$ $\quad$ $b^2 = 2a - 2b$

$a = 3b$

$b^2 = 4b$

$b = 4$, $a = 12$

$a - b = \mathbf{8}$

F02B5. The coefficient is ${}_6C_3 * 2^3(-3)^3 = 20 * 8 * (-27) = \mathbf{-4320}$

F02B6. $x^2 + 3x + 2 = (x+1)(x+2)$. Since either $x+1$ or $x+2$ must be even, $x^2 + 3x + 2$ must be divisible by 2. The only time $(x+1)(x+2)$ will not be divisible by 3 is when x is divisible by 3. Of the 90 2-digit numbers, 30 are multiples of 3. $90 - 30 = \mathbf{60}$.

SOLUTIONS

F02B7. $\log_2\left(\dfrac{128\sqrt{8}}{\sqrt[3]{32}}\right) = \log_2(128) + \log_2\left(\sqrt{8}\right) - \log_2\left(\sqrt[3]{32}\right) = 7 + \dfrac{3}{2} - \dfrac{5}{3} = \dfrac{41}{6}$

F02B8. $p + q = {-b}/{a}$ $\quad pq = {c}/{a}$

$$\frac{1}{p^2} + \frac{1}{q^2} = \frac{p^2 + q^2}{(pq)^2} = \frac{(p+q)^2 - 2pq}{(pq)^2} = \frac{\left(-b/a\right)^2 - 2c/a}{\left(c/a\right)^2} = \frac{b^2 - 2ac}{c^2}$$

F02B9. Let $F(x, y) = (x - y)^{10}$, then the sum of the coefficients would be equivalent to $F(1,1) = (1-1)^{10} = 0$.

F02B10. Let the legs of the right triangle be 2a and 2b. Then $37 = 4a^2 + b^2$, $38 = a^2 + 4b^2$, and $75 = 5(a^2 + b^2)$.

Hypotenuse = $\sqrt{4a^2 + 4b^2} = \sqrt{75\left(4/5\right)} = \sqrt{60} = 2\sqrt{15}$

F02B11. If $x^2 - 5x + 6 = 0$, then $x = 2$ or $x = 3$

If $x + 3 = 1$, then $x = -2$

If $x + 3 = -1$, then $x = -4$ checks since $x^2 - 5x + 6$ is always even

Therefore the values of x are **-4, -2, 2, 3**

F02B12. Solution 1 Theorem: $(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$

$(PD)^2 = (PA)^2 + (PC)^2 - (PB)^2 = 16 + 64 - 36 = 44$

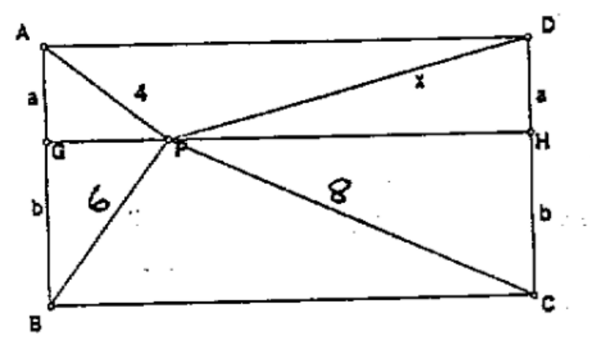
$PD = \sqrt{44} = 2\sqrt{11}$

Solution 2 Through point P, draw a line parallel to AD.

$4^2 - a^2 = 6^2 - b^2$

$x^2 - a^2 = 8^2 - b^2$

$x^2 - 4^2 = 8^2 - 6^2$ and therefore $x = \sqrt{44} = 2\sqrt{11}$



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER THREE** **Fall 2002**

SOLUTIONS

F02B13. Soln1: 109 = 64+32+8+4+1, hence, $\mathbf{1101101_2}$

Soln2: Divide 109 and each resulting quotient by 2 continuously and take the remainders in reverse order:

54R1, 27R0, 13R1, 6R1, 3R0, 1R1, 0R1 $\mathbf{1101101_2}$

Soln3: Notice that $109 = 127 - 18 = 127 - 16 - 2 = 1111111_2 - 10000_2 - 10_2 = \mathbf{1101101_2}$

F02B14. $2x = 2002 - 3y$, so $2002 - 3y$ must be a positive even number. So y can be any even number from 2 to 666, making a total of **333** solutions.

F02B15. Soln1: There are 45 of them, from 10 to 98: $\frac{45(10+98)}{2} = \mathbf{2430}$

Soln2: let $x = 98 + 96 + \ldots + 10$, then $x + 20 = 98 + 96 + \ldots + 0 = 50*98/2 = 2450$

$x = 2450 - 20 = \mathbf{2430}$

F02B16. Turn it into the sum of 2 geometric progressions:

$$\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\ldots=\frac{1/5}{1-1/5}=\frac{1}{4} \qquad \frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\ldots=\frac{1/25}{1-1/25}=\frac{1}{24}$$

$$\frac{1}{4}+\frac{1}{24}=\frac{7}{24}$$

F02B17. $x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2=3-2=1$

$$\left(x^2+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)=x+\frac{1}{x}=x^3+x+\frac{1}{x}+\frac{1}{x^3}=x^3+\frac{1}{x^3}+x+\frac{1}{x}$$

$$0=x^3+\frac{1}{x^3}$$

F02B18. Soln1:

$(x+y+z)^{10} = ([x+y]+z)^{10} = (x+y)^{10} + {}_{10}C_1\ (x+y)^9 z + {}_{10}C_2\ (x+y)^8 z^2 + \ldots z^{10}$

Which has 11 + 10 + 9 + … 1 = **66** terms

Soln2: $(x+y+z)$ has 3 term

$(x+y+z)^2$ has 6 terms

$(x+y+z)^3$ has 10 terms

Continuing this pattern: 3, 6, 10, 15, 21, 28, 36, 45, 55, **66**

This sequence of numbers is also called triangular numbers.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

SENIOR B DIVISION **CONTEST NUMBER FOUR** **Fall 2002**

SOLUTIONS

F02B19. $2^0 = \log_3(\log_4 x) = 1, \qquad 3^1 = \log_4 x = 3, \qquad 4^3 = x = \mathbf{64}$

F02B20. If they work x minutes, a postal worker can do $\frac{x}{30}$ of the job while the machine can do $\frac{x}{20}$.

$1 = 4\left(\frac{x}{20}\right) + 3\left(\frac{x}{30}\right) = \frac{x}{5} + \frac{x}{10} = \frac{3x}{10}, \ x = \frac{10}{3}$ minutes.

F02B21. Let the common radius of the semicircle and the circle be r, then

$r^2 = \left(\sqrt{10}\right)^2 + \left(\sqrt{10}/2\right)^2 = 10 + \frac{10}{4} = \frac{25}{2}$

Therefore, the diameter of circle P is $5\sqrt{2}$. Since we know that the product of the diagonals over 2 is the area of a square, then the area is just $\frac{\left(5\sqrt{2}\right)^2}{2} = \frac{50}{2} = 25$

F02B22. Any number with a factor of 5 times any even number will produce a 0. Since there are many even numbers, we need to count the number of factors of 5, and since 25 and 50 have 2 each, the total is 12.

F02B23. The sum of the test marks is 738. Subtracting 50, $688/8 = 86$.

F02B24. The minute hand moves 12 times as fast as the hour hand, and moves $12X$ degrees to the hour hand's X degrees.

$12x - x = 90$

$x = 90/11$

$12x = 12 \cdot 90/11$

Since every 6 degrees on the minute hand is one minute, the number of minutes after 12 o'clock is

$\frac{12 \cdot 90}{6 \cdot 11} = \frac{180}{11} = 16\frac{4}{11} = 16$ minutes 22 seconds to the nearest second.

The answer is **(16, 22)**.

SOLUTIONS

F02B25. $\frac{600^2}{(301+299)(301-299)} = \frac{600^2}{600 \cdot 2} = 300$

F02B26. $2002 = 2 \cdot 7 \cdot 11 \cdot 13$ The number of factors is computed by adding one to each of the exponents and taking the product, or $2 \cdot 2 \cdot 2 \cdot 2 = 16$

F02B27. All 3 balls must be odd. Since there are 8 odd balls, the probability is.
$\frac{8}{15} \cdot \frac{7}{14} \cdot \frac{6}{13} = \frac{8}{65}$
or
the probability is $\frac{{}_8C_3}{{}_{15}C_3} = \frac{56}{455} = \frac{8}{65}$

F02B28. The volume of a pyramid is $\frac{1}{3}hA$. The area of the base $= \frac{s^2}{4}\sqrt{3} = \frac{9}{4}\sqrt{3}$
The height is one leg of a right triangle, the hypotenuse being an edge, 3, and the other leg being the distance from the center of the equilateral triangle to a vertex, $\sqrt{3}$

$$\left(\sqrt{3}\right)^2 + h^2 = 3^2$$

$$h = \sqrt{6}$$

$$V = \frac{1}{3} \cdot \frac{9}{4} \cdot \sqrt{3} \cdot \sqrt{6} = \frac{3}{4}\sqrt{18} = \frac{9}{4}\sqrt{2}$$

F02B29. Since $BF = 2FD$, the area of $\triangle BFC = \frac{2}{3}$ area of $\triangle CBD$

$$\frac{2}{3} \cdot \frac{1}{2} \cdot 25\sqrt{3} = \frac{25}{3}\sqrt{3}$$

F02B30. The number is divisible by 9. Therefore $38+a+b$ is a multiple of 9, and $a+b = 7$ or $a+b = 16$.
Also, the number is divisible by 11. Therefore $26 + a$ and $12 + b$ differ by a multiple of 11. Either $26 + a = 12 + b + 11$ or $26 + a = 12 + b + 11 + 11$. The first gives $b - a = 3$, the second $a - b = 8$.
Combining $a + b = 7$ and $b - a = 3$ gives the only single digit positive integral solution of $a = 2$ and $b = 5$. The answer is **(2, 5)**.